Equivalent delta-v per orbit of gravitational perturbations*

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Nomenclature

a = semi-major axis (orbital element); length unit
C = subindex for a term derived from the Coriolis effect
c = subindex for a term derived from the centrifugal forces
D = subindex for disturbing function
e = eccentricity (orbital element); dimensionless
f = true anomaly; rad
$\boldsymbol{G}$ = angular momentum vector per unit of mass; $(\mathrm{length\,unit})^2/\mathrm{time\,unit}$
I = inclination (orbital element); rad
$\boldsymbol{J}$ = impulse per unit of mass (delta-v); length unit/time unit
J_2 = zonal harmonic coefficient of the second degree; dimensionless
K = subindex for Keplerian
$\boldsymbol{k}$ = unit vector in the z direction; dimensionless
M = mean anomaly; rad
$\boldsymbol{N}$ = reference system's rotation rate; rad/time unit
N = modulus of $\boldsymbol{N}$; rad/time unit
n = orbit mean motion $n=\sqrt{\mu/a^3}$; rad/time unit
p = conic parameter $p=a\eta^2$; length unit

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q = auxiliary symbol of order 1; dimensionless
$\boldsymbol{r}$ = radius from the earth's center of mass; length unit
r = distance from the earth's center of mass; length unit
S = scalar part of the impulse in the rotating frame; length unit/time unit
t = time
V = gravity potential; $(\text{length unit})^2/(\text{time unit})^2$
x, y, z = Cartesian coordinates of $\boldsymbol{r}$; length unit
α = earth's equatorial radius; length unit
$\Delta\boldsymbol{v}$ = delta-v (impulse per unit of mass); length unit/time unit
ϵ = auxiliary symbol of the order of the perturbation; dimensionless
η = eccentricity function $\eta = \sqrt{1-e^2}$; dimensionless
θ = argument of latitude $\theta = f + \omega$; rad
μ = earth's gravitational parameter; $(\text{length unit})^3/(\text{time unit})^2$
ρ, σ = scalar indices representing the perturbed dynamics; dimensionless
τ = time of perigee passage (orbital element); time unit
Ω = right ascension of the ascending node (orbital element); rad
ω = argument of the perigee (orbital element); rad
$*$ = superindex. The indexed magnitude is valid under the averaging assumption

1 Introduction

It is demonstrated that the impulse in a single orbit of an earth's satellite can be computed analytically within a reasonable accuracy, in this way providing a vectorial expression that is used to reveal the more important features of the earth's satellite dynamics without need of resorting to Fourier series expansions or perturbation theory. The impulse is referred to a rotating frame, and is computed in the "averaging assumption", that is, taking the osculating semi-major axis, eccentricity, and inclination constant in the short time interval in which the argument of the latitude of the satellite advances by 2π. In particular, by analyzing the impulse due to the perturbation that arises from the J_2 term of the earth's gravitational potential it is shown the existence of planar orbits, which may exist in the equatorial plane and also in the meridian planes (polar orbits), as well as orbits with fixed perigee at the critical inclination —all of them in agreement with well-known results.

In addition, it is shown that the information provided by this impulse can be encapsulated in simple scalar indices. The evaluation of these kinds of indices can be used for the creation of argument of the perigee vs. inclination maps, which provide valuable information on the perturbed dynamics. Furthermore, the procedure is not restricted either to the J_2 perturbation or to conservative perturbations, and can be generally applied, although in most cases the quadratures defining the impulse should be evaluated using numerical integration techniques.

The impulse is computed per unit of mass, which therefore will produce a corresponding variation of the velocity vector instead of the linear momentum. Hence, it provides the equivalent delta-v per orbit of gravitational perturbations, which can be useful in practical engineering problems. For instance, some maneuvers are designed using just the Keplerian approximation [1]. Also, the initial design of constellations is commonly made under the linear effect of the J_2 perturbation, which does not include the effects of the perigee dynamics [2]. In both cases a better estimation of the delta-v budget necessary to compensate the neglected effects may be obtained from the theory provided here.

2 Δv on a Keplerian orbit

The net delta-v from Keplerian motion between two times is

$$\Delta \boldsymbol{v} = \int_{t_1}^{t_2} \left(-\frac{\mu}{r^3} \boldsymbol{r} \right) \mathrm{d}t, \tag{1}$$

where the time dependence of $\boldsymbol{r}$ must be made explicit to perform the quadrature. This can be done using the orbital elements representation of $\boldsymbol{r}$.

$$\frac{x}{r} = \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos I, \tag{2}$$

$$\frac{y}{r} = \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos I, \tag{3}$$

$$\frac{z}{r} = \sin\theta\sin I, \tag{4}$$

where

$$r = \frac{p}{1 + e\cos f}, \tag{5}$$

Equations (2)–(4) and (5) reveal the time dependence of the Cartesian coordinates of the Kepler problem, but this is done in an implicit way through the true anomaly. Although the true anomaly is an implicit function of time, the quadrature in Eq. (1) can be solved directly either in f or θ making use of the preservation of the total angular momentum of the Keplerian motion, which can be stated in the usual form

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \|\boldsymbol{G}\| = \sqrt{\mu\, p} = n a^2 \eta. \tag{6}$$

where $\mathrm{d}t = (1/n)\,\mathrm{d}M$, and hence

$$n\,\mathrm{d}t = \mathrm{d}M = \frac{r^2}{a^2\eta}\,\mathrm{d}\theta, \tag{7}$$

making in this way available an explicit relation between the differentials of the mean anomaly and the argument of the latitude.

An analytical expression of the $\Delta\boldsymbol{v}$ of Keplerian motion between two arbitrary times is obtained substituting Eq. (7) into Eq. (1) to give

$$\Delta\boldsymbol{v} = -\frac{na}{\eta}\int_{\theta(t_1)}^{\theta(t_2)} \frac{\boldsymbol{r}}{r}\,\mathrm{d}\theta, \tag{8}$$

which is integrated replacing $\boldsymbol{r}/r$ by its components in Eqs. (2)–(4), namely

$$\Delta\boldsymbol{v} = \frac{na}{\eta}\left(\begin{array}{c} -\sin\theta\cos\Omega - \cos I\cos\theta\sin\Omega \\ -\sin\theta\sin\Omega + \cos I\cos\theta\cos\Omega \\ \sin I\cos\theta \end{array}\right)\Bigg|_{\theta(t_1)}^{\theta(t_2)}. \tag{9}$$

As expected, Eq. (9) vanishes when it is evaluated along a full orbit $\theta(t_2)-\theta(t_1) = 2\pi$.

3 Effects of J_2

For the earth, it happens that $J_2 = \mathcal{O}(10^{-3})$ whereas all other coefficients are of order 10^{-6}. Then, the non-central potential

$$V = -\frac{\mu}{r} + \frac{\mu}{r}\frac{\alpha^2}{r^2}J_2\frac{1}{2}\left(-1 + 3\frac{z^2}{r^2}\right), \tag{10}$$

is quite representative of a wide class of non-resonant low earth orbits.

The Newtonian equations of motion derived from Eq. (10) are written in the vectorial form

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{\mu}{r^3}\boldsymbol{r} + 3\frac{\mu}{r^3}\frac{\alpha^2}{r^2}J_2\left[\left(\frac{5}{2}\frac{z^2}{r^2} - \frac{1}{2}\right)\boldsymbol{r} - z\boldsymbol{k}\right], \tag{11}$$

whose solutions are no longer ellipses. However, as far as J_2 is small, solutions to Eq. (11) may be viewed as slightly distorted ellipses whose deformation evolves with time, a case in which the concept of *osculating* elements can be used in the description of the non-Keplerian motion. Thus, the position and velocity of the body at a given moment in time can be used to define an instantaneous ellipse that is tangent (osculating) to the actual trajectory. Since each instantaneous ellipse is defined by corresponding Keplerian elements, the non-Keplerian trajectory can be described by the time evolution of these osculating elements, to whom all the relations of the elliptic motion apply.

The osculating elements representation is quite useful to manifest the time dependency of Eq. (11), in this way enabling the computation of the corresponding $\Delta\boldsymbol{v}$ under the averaging assumption. That is, the osculating elements $a(t)$, $e(t)$, $I(t)$, $\Omega(t)$, $\omega(t)$, and $\tau(t)$ are assumed to evolve slowly when compared to the fast evolution of either the mean anomaly, which is driven by the osculating mean motion $n(t)$. Since the rate of variation of the argument of the latitude will only differ from n in effects of the order of the perturbation (J_2 in the present case), which are related to the non-vanishing motion of argument of the perigee, these slow varying elements are taken as constants in the computation of the quadratures defining the $\Delta\boldsymbol{v}$ in the interval in which θ increases by 2π, that is, a full orbit.

3.1 The rotating frame

Note, however, that while taking a, e, and I constant along a full orbit may be a reasonable approximation, the assumption that ω and Ω remain constant in the same short time interval must be further qualified. In particular, as far as the right ascension of the ascending node may have a slightly different value in the initial and final times, independently of the smallness of this a priori unknown variation, taking Ω constant is equivalent to say that the quadratures are solved in a *rotating* frame which is moving with the rotation rate of Ω, a case in which corresponding inertia terms should be added to obtain the $\Delta\boldsymbol{v}$.

The acceleration in the rotating frame is

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} + 3\frac{\mu}{r^3}\frac{\alpha^2}{r^2}J_2\left[\left(\frac{5}{2}\frac{z^2}{r^2} - \frac{1}{2}\right)\boldsymbol{r} - z\boldsymbol{k}\right] - 2\boldsymbol{N}\times\dot{\boldsymbol{r}} - \boldsymbol{N}\times(\boldsymbol{N}\times\boldsymbol{r}), \quad (12)$$

where $\boldsymbol{N} = N\boldsymbol{k}$ is the rotation rate of the node, which is further assumed to remain constant in the averaging assumption, and overdots are used to emphasize that corresponding derivatives are measured in the rotating frame.

Then, the $\Delta\boldsymbol{v}$ in the rotating frame is written

$$\Delta\boldsymbol{v} = \boldsymbol{J}_{\mathrm{K}} + \boldsymbol{J}_{\mathrm{D}} + \boldsymbol{J}_{\mathrm{C}} + \boldsymbol{J}_{\mathrm{c}}, \quad (13)$$

where the Keplerian part $\boldsymbol{J}_{\mathrm{K}}$ is formally the same as in Eq. (1), viz.

$$\boldsymbol{J}_{\mathrm{K}} = \int_{t_1}^{t_2}\left(-\frac{\mu}{r^3}\boldsymbol{r}\right)\mathrm{d}t, \quad (14)$$

but now $\boldsymbol{r}$ no longer corresponds to Keplerian motion. The part of $\Delta\boldsymbol{v}$ due to the J_2 perturbation is

$$\boldsymbol{J}_{\mathrm{D}} = \int_{t_1}^{t_2} 3\frac{\mu}{r^3}\frac{\alpha^2}{r^2}J_2\left[\left(\frac{5}{2}\frac{z^2}{r^2}-\frac{1}{2}\right)\boldsymbol{r}-z\boldsymbol{k}\right]\mathrm{d}t, \tag{15}$$

that of the Coriolis acceleration is

$$\boldsymbol{J}_{\mathrm{C}} = \int_{t_1}^{t_2}(-2\boldsymbol{N}\times\dot{\boldsymbol{r}})\,\mathrm{d}t, \tag{16}$$

and the part of $\Delta\boldsymbol{v}$ of the centrifugal terms is

$$\boldsymbol{J}_{\mathrm{c}} = \int_{t_1}^{t_2}[-\boldsymbol{N}\times(\boldsymbol{N}\times\boldsymbol{r})]\,\mathrm{d}t. \tag{17}$$

3.2 $\Delta\boldsymbol{v}$ along a full orbit

The quadratures in Eqs. (14)–(17) can be solved in closed form in the averaging assumption. In the rotating frame, taking into account Eqs. (2)–(4) and in view of the differential relation in Eq. (7) also applies to the osculating case, it is easily found

$$\boldsymbol{J}_{\mathrm{K}}^{*} = -\frac{na}{\eta}\int_{\theta_1}^{\theta_1+2\pi}\frac{\boldsymbol{r}}{r}\,\mathrm{d}\theta = \boldsymbol{0}, \tag{18}$$

where the asterisk notation is used to indicate that the quadrature is solved in the averaging assumption. Note that, because the full orbit has been defined in terms of the argument of the latitude rather than the mean anomaly, dealing with the (unknown) motion of the perigee is avoided in the integration of Eq. (18). Therefore, the $\Delta\boldsymbol{v}$ undergone by the satellite along a full orbit is due only to the J_2 perturbation and the inertia terms.

Because the flow represented by Eq. (11) enjoys axial symmetry, following derivations are simplified without loss of generality by choosing $\Omega = 0$. Then, Eq. (15) is rewritten as

$$\boldsymbol{J}_{\mathrm{D}} = \frac{3}{2}J_2\int_{\theta_1}^{\theta_2}\frac{na}{\eta}\frac{\alpha^2}{r^2}\begin{pmatrix}(1-5\sin^2 I\sin^2\theta)\cos\theta\\ \cos I(1-5\sin^2 I\sin^2\theta)\sin\theta\\ \sin I(3-5\sin^2 I\sin^2\theta)\sin\theta\end{pmatrix}\mathrm{d}\theta, \tag{19}$$

where the dependence of r on the argument of latitude is made explicit by replacing $f = \theta - \omega$ in Eq. (5). Because $\boldsymbol{J}_{\mathrm{D}}$ is already of the order of J_2, errors introduced by neglecting time variations of ω when integrating Eq. (19) along a full orbit are of higher order of J_2. Then, Eq. (19) is integrated between θ_1 and $\theta_1 + 2\pi$ to give

$$\boldsymbol{J}_{\mathrm{D}}^{*} = S \begin{pmatrix} (1 - 5\cos^2 I)\cos\omega \\ (11 - 15\cos^2 I)\,\cos I \sin\omega \\ 3(1 - 5\cos^2 I)\,\sin I \sin\omega \end{pmatrix}, \tag{20}$$

where the scalar part

$$S = \frac{3}{4} J_2 \pi a n \frac{\alpha^2}{p^2}\frac{e}{\eta} = \left(\frac{3}{4} J_2 \pi \sqrt{\frac{\mu}{\alpha}}\right) \frac{1}{(a/\alpha)^{5/2}} \frac{e}{\eta^5}, \tag{21}$$

is made of the product of a constant part related to the parameters of the problem, enclosed by brackets in the right side of Eq. (21), and a nondimensional part which varies with the orbit semi-major axis and eccentricity. On the other hand, the vectorial part of $\boldsymbol{J}_{\mathrm{D}}^{*}$ only depends on the inclination and the argument of the perigee.

Therefore, if the variations experienced by the orbital elements as a result of the disturbing effect of J_2 are small, then, when the argument of the latitude increases by 2π the satellite undergoes a variation of its velocity in the rotating frame as given by Eq. (20). Note that, even though the argument of the perigee is poorly defined for the lower eccentricity orbits, Eq. (20) still applies by using the elements $e\cos\omega$ and $e\sin\omega$, instead of e and ω.

Besides, since the precession of the node is a consequence of the J_2 perturbation, it is further assumed that $(N/n) = \mathcal{O}(J_2)$. Hence, carrying out the quadratures related to the inertia forces either in the mean anomaly or the argument of the latitude will only differ in higher order of J_2 terms. Then, it is simple to check that

$$\boldsymbol{J}_{\mathrm{C}}^{*} = \int_{t(M_1)}^{t(M_1+2\pi)} (-2\boldsymbol{N} \times \dot{\boldsymbol{r}})\,\mathrm{d}t = \boldsymbol{0}, \tag{22}$$

thus making null the contribution of the Coriolis acceleration to $\Delta\boldsymbol{v}^*$ in the rotating frame. Finally,

$$\boldsymbol{J}_{\mathrm{c}}^{*} = \int_{t(M_1)}^{t(M_1+2\pi)} [-\boldsymbol{N} \times (\boldsymbol{N} \times \boldsymbol{r})]\,\mathrm{d}t = -3\pi a n e \frac{N^2}{n^2} \begin{pmatrix} \cos\omega \\ \cos I \sin\omega \\ 0 \end{pmatrix}, \tag{23}$$

provides the part of $\Delta\boldsymbol{v}^*$ due to the centrifugal forces.

Therefore, the total $\Delta\boldsymbol{v}^*$ in a full orbit and under the averaging assumption is obtained by adding Eqs. (20) and (23), to give

$$\Delta\boldsymbol{v}^* = S \begin{pmatrix} (1 - 5\cos^2 I - \epsilon)\cos\omega \\ (11 - 15\cos^2 I - \epsilon)\cos I \sin\omega \\ 3(1 - 5\cos^2 I)\sin I \sin\omega \end{pmatrix}, \tag{24}$$

where

$$\epsilon = 4\frac{(N/n)^2\eta}{J_2(\alpha/p)^2}, \tag{25}$$

which is positive for the earth in view of it has the same sign of J_2.

Note that, because $\epsilon = \mathcal{O}(J_2)$, the contribution of the centrifugal acceleration to the $\Delta\boldsymbol{v}$ might be neglected in the 2π interval for small values of J_2. Hence, working at first order of J_2, Eq. (20) still applies to a rotating frame whose average rotation rate is

$$N = [\Omega(t_2) - \Omega(t_1)]\,\frac{n}{2\pi} = n\,\mathcal{O}(J_2). \tag{26}$$

3.3 Discussion

Equation (20) shows that, up to first order effects of J_2, the qualitative effects of the dynamics only depend on the orbit inclination and argument of the perigee, whereas the eccentricity and semi-major axis just scale the problem. Because of this, the lower eccentricity orbits undergo the smaller perturbations for a given semi-major axis. Conversely, for a given eccentricity the lower altitude orbits experience the higher perturbations. In particular, circular orbits remain circular in the rotating frame. This behavior is illustrated in Fig. 1, where the nondimensional, non-constant part of S, viz. $e((1-e^2)(a/\alpha))^{-5/2}$, is represented for the variation of the semi-major axis between one and five earth's radius and all the range of eccentricities.

Other dynamical features of the J_2 problem are derived from the vectorial part of Eq. (20). Thus, when $\cos I = 0$ the y axis component of $\boldsymbol{J}_{\mathrm{D}}^{*}$ vanishes. Since the x axis has been chosen in the direction of the ascending node, it happens that polar orbits are planar solutions of the J_2 problem. In this case, the magnitude of the vector in Eq. (20) is $\sqrt{5-4\cos 2\omega}$, showing that the less perturbed polar orbits have $\omega = 0$ and the more perturbed polar orbits occur with $\omega = \pi/2$. Besides, the z axis component of $\boldsymbol{J}_{\mathrm{D}}^{*}$ vanishes for equatorial orbits, so they are planar solutions where the non-vanishing x and y components of $\boldsymbol{J}_{\mathrm{D}}^{*}$ evolve harmonically with constant modulus. Finally, for those critical inclinations such that $\cos^2 I = 1/5$, the variation of the velocity vector in one full orbit takes the direction of the y axis (the direction orthogonal to the node) and is proportional to $\sin\omega$. Hence, in the rotating frame, orbits with equatorial perigee do not experience any variation of the velocity vector in a full orbit. For non-equatorial perigees the y axis component of $\boldsymbol{J}_{\mathrm{D}}^{*}$ no longer vanishes, but it is always possible to find a different rotating frame where it does. Therefore, orbits with critical inclination are closed (non-Keplerian) ellipses in the right rotating frame.

Second order corrections due to the centrifugal forces introduce both qualitative and quantitative variations. Namely, the appearance of $\epsilon > 0$ in Eq. (24)

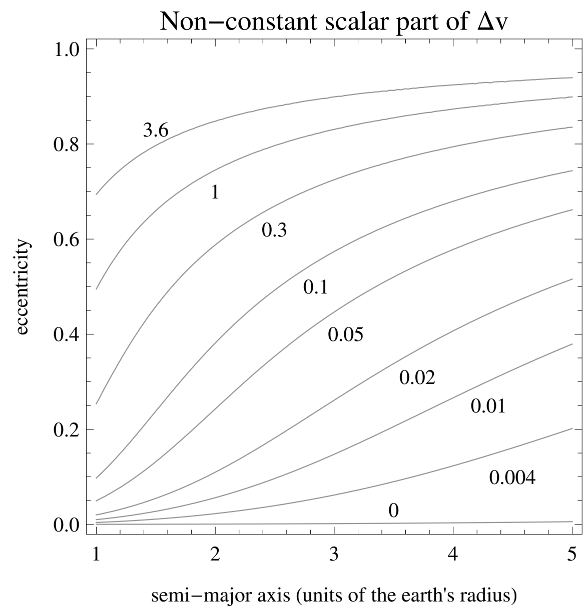

Figure 1: Variation of the non-constant part of S in Eq. (21). Nondimensional units.

prevents the simultaneous vanishing of the components of $\Delta\boldsymbol{v}^*$ in the x and z axes directions except for orbits with $\omega = \pm\pi/2$ and critical inclination, and orbits with equatorial perigee and such inclination that $5\cos^2 I + \epsilon = 1$. Hence, closed non-Keplerian ellipses in the rotating frame can occur only in these two cases. This is an important modification with respect to the first order dynamics, where all critically inclined orbits undergo zero $\Delta\boldsymbol{v}^*$. Finally, because ϵ depends on p and η, in addition to scaling $\Delta\boldsymbol{v}^*$, the eccentricity and semi-major modify the magnitude of ϵ, but these changes do not modify substantially the dynamics. Note that ϵ also depends on N, which in turn may depend on e and a; however, the possible appearance of e and a in N is not expected to cancel, in general, the term $p^2\eta$ of ϵ.

4 Searching for a significant scalar index

From the discussion above, it seems natural to define a scalar index based on the magnitude of the $\Delta\boldsymbol{v}$ of the disturbing force in a full orbit. This kind of index provides a measure of the variation of the velocity vector when θ has increased by 2π, and will be general for any perturbed problem, either conservative or not.

Note that this definition is essentially different from a similar one in [3], which provides an estimation on how the perturbed orbit departs from the Keplerian case by simply accumulating the magnitude of the disturbing acceleration when the mean anomaly increases by 2π, thus missing the fundamental cases in which the perturbation effect may balance along the orbit. The complementary use of both kinds of indices may deserve further study.

4.1 First order effects

In the particular case of the J_2 problem, the significant information is obtained from the non-dimensional index

$$\rho = \sqrt{(1 - 5\cos^2 I)^2 + 8\sin^2 I(1 + 5\cos^2 I)\sin^2 \omega}, \tag{27}$$

which will give an approximation of the active control

$$\Delta v = \|\boldsymbol{J}^*_{\mathrm{D}}\| = S\rho, \tag{28}$$

as derived from Eq. (20), needed for the maintenance of a nominal, non-Keplerian orbit under the J_2 perturbation.

In view of Eq. (27) does not depend on any parameter, the qualitative features of the J_2 perturbed dynamics can be represented by a single contour plot in the (ω, I) plane. Application to different orbital configurations in the plane (a, e) of dynamical parameters becomes, then, a simple matter of scaling.

The evolution of the index ρ is presented in Fig. 2, where, for the symmetries of Eq. (27), the range is limited to direct orbits and $-\pi/2 \le \omega \le \pi/2$. As shown in the figure, apart from equatorial orbits where the index takes the constant value $\rho = 4$, the variations in the velocity vector undergone by the satellite in a full orbit always increase when the argument of the perigee varies from 0 to $\pm\pi/2$ and generally decrease with inclination. There are two notable exceptions to this general behavior. One the one hand, as expected from the previous discussion above, Δv vanishes at the critical inclination when the argument of the perigee is 0. On the other hand, the maximum Δv occurs when $\omega = \pm\pi/2$ and the inclination is about $30\,\mathrm{deg}$. More precisely, as derived from Eq. (27), this maximum takes the value $\rho = \sqrt{256/15} \approx 4.13$, and happens when $I = \arccos\sqrt{11/15} \approx 31.1\,\mathrm{deg}$. Remarkably, any orbit with this last inclination has null component of $\boldsymbol{J}^*_{\mathrm{D}}$ in the y axis direction.

Note that the value to which the index ρ evaluates may obscure some relevant information on the orbit behavior. Indeed, a control strategy based on Eq. (27) includes efforts in minimizing the departure from the node in the rotating frame.

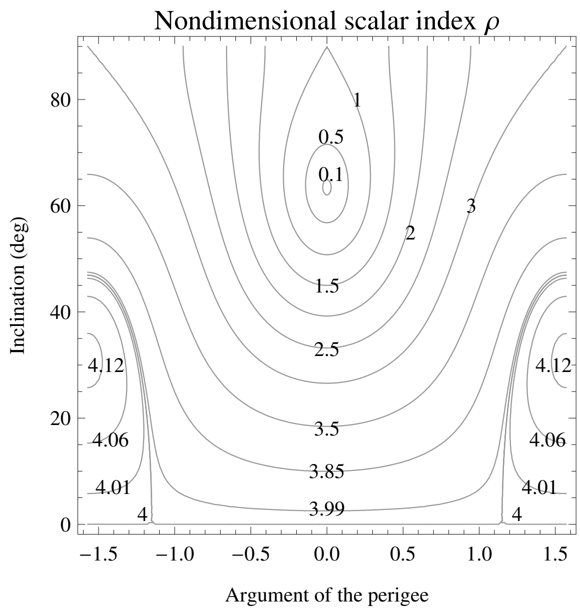

Figure 2: Variation of the nondimensional index $\rho \equiv \rho(\omega, I)$ in Eq. (27).

However, increasing the control trying to minimize this departure may be unpractical, and common design strategies may focus on the orbit evolution in the orbital plane. Hence, an alternative scalar index that neglects the y axis component of the variation of the velocity vector in the rotating frame may be considered. Thus, the new scalar, nondimensional index

$$\sigma_{\rm D} = \left|1 - 5\cos^2 I\right| \sqrt{1 + (8 - 9\cos^2 I)\sin^2\omega}, \tag{29}$$

is defined by neglecting the y component of $\boldsymbol{J}_{\rm D}^*$ in Eq. (20).

A contour plot based on the index $\sigma_{\rm D}$ shows interesting details that remained hidden in Fig. 2. Now, as displayed in Fig. 3, $\sigma_{\rm D}$ vanishes for any orbit with critical inclination regardless of its argument of the perigee, as noted by the horizontal contour in the upper part of Fig. 3, in agreement with the well-known behavior predicted by the secular theory up to the first order, cf. Eq. (10.95) of [4], for instance. Other differences with respect to the ρ diagram in Fig. 2 arise in the region of low inclination orbits, where $\sigma_{\rm D}$ vanishes for $I = 0$ and $\omega = \pm\pi/2$. Nevertheless, except for values of ω close to $\pm\pi/2$, the lower inclination orbits still show as highly perturbed in terms of $\sigma_{\rm D}$. On the other hand, with the definition of the new index the most perturbed orbits remain at the inclination $I = \arccos\sqrt{11/15}$; this was expected because they happen at $\omega = \pm\pi/2$ where the neglected y component of $\boldsymbol{J}_{\mathrm{D}}^{*}$ vanishes, and hence the corresponding value of the index is not affected: $\sigma = \sqrt{256/15} \approx 4.13$. Finally, all the orbits with $I = \arccos\sqrt{8/9} \approx 19.5\,\mathrm{deg.}$ undergo a $\Delta\boldsymbol{v}$ with the same magnitude, as represented by σ_{D}, irrespective of their argument of the perigee, as noted by the horizontal contour in the lower part of Fig. 3.

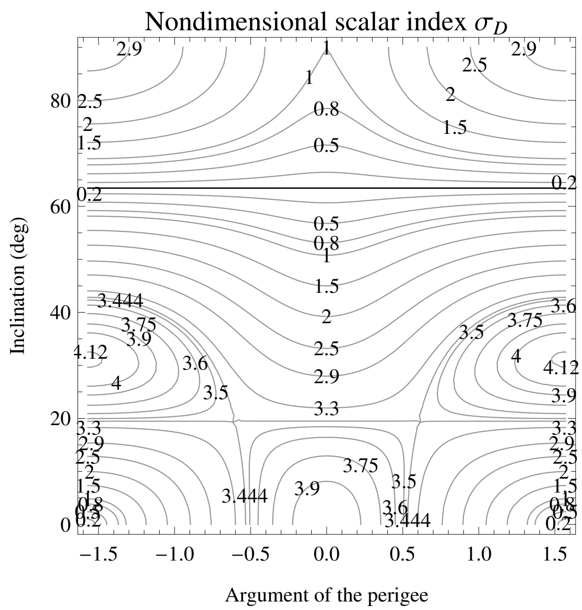

Figure 3: Contour levels $\sigma_{\mathrm{D}} \equiv \sigma_{\mathrm{D}}(\omega, I)$ as defined in Eq. (29).

4.2 Second order corrections

The index σ_{D} can be further refined to include second order effects due to the centrifugal forces. Then, the full definition of $\Delta\boldsymbol{v}^{*}$ in Eq. (24) with the y axis component removed is used to define the scalar index

$$\sigma = \sqrt{(1 - 5\cos^2 I - \epsilon)^2 \cos^2\omega + 9\,(1 - 5\cos^2 I)^2 \sin^2 I \sin^2\omega}, \tag{30}$$

which now, because of the form of ϵ in Eq. (25) includes the dependency on a and e in the index, loosing in this way the generality of the indices ρ and σ_{D} used before.

For a qualitative description based on σ it is enough to know that ϵ is positive and of the order of J_2. Then, in order to illustrate the refinements introduced by the new index in the description of the dynamics, a value $\epsilon = J_2 = 10^{-3}$ has been chosen as representative of the earth's case. By taking this fixed value, the dependency on the eccentricity and semi-major axis is formally removed from σ, which, therefore, remains general for any a and e. Now, as shown in Fig. 4, the new index is able to reveal the modifications of the dynamics introduced by the second order effects of J_2, showing that only two orbits survive with $\sigma = 0$, either with equatorial argument of the perigee or with $\omega = \pi/2$. For the former, the inclination slightly departs from the critical value in agreement with the previous discussion in Section 3.3.

Note that, since the modifications introduced by σ are of the order of J_2, the values of the contours displayed in Fig. 4, and also in the next Fig. 5, have been multiplied by 1000 in order to get the plots as legible as possible.

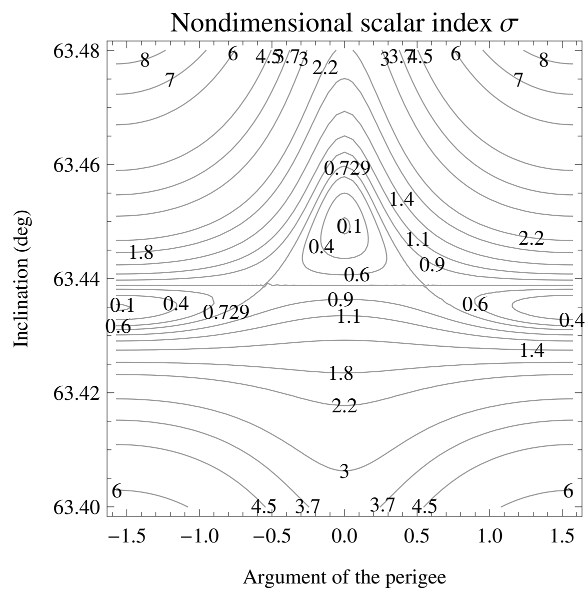

Figure 4: Contours σ, as given by Eq. (30) with $\epsilon = 10^{-3}$, in the vicinity of the critical inclination.

A more precise index can be obtained by making

$$N = -\frac{3}{2} J_2 \frac{\alpha^2}{p^2} n \cos I, \tag{31}$$

an expression that corresponds to the secular rate of the classical first order approach based on the Lagrange planetary equations —cf. Eq. (10.94) of [4]. Then, plugging Eq. (31) into Eq. (25) results in

$$\epsilon = 9 J_2 \frac{\alpha^2}{p^2} \eta \cos^2 I, \tag{32}$$

and hence

$$\sigma = \sqrt{(1 - 5q\cos^2 I)^2 \cos^2\omega + 9\,(1 - 5\cos^2 I)^2 \sin^2 I \sin^2\omega}, \tag{33}$$

where the modifier

$$q = 1 + \frac{9}{5} J_2 \frac{\alpha^2}{a^2} \frac{1}{\eta^3}, \tag{34}$$

now includes the expected dependency of σ on the orbit semi-major axis and eccentricity, thus requiring different contour plots for different orbital configurations in the (a, e) plane. This effect is only quantitative and can be appreciated in Fig. 5.

This final refinement of the scalar index would require to complement the present approach with the classical theory, which seems to be unnecessary at all from the point of view of providing a qualitative description of the J_2 perturbed dynamics.

5 Conclusions

The physical definition of the delta-v provides a simple way of investigating the dynamics of perturbed Keplerian problems. In the case of the dynamics arising from the J_2 term of the earth's gravitational potential, the delta-v of the disturbing function in the time interval in which the argument of the latitude increases by 2π has been computed analytically, from which two different scalar indices have been derived. In particular, the dynamics of orbits at the critical inclination is correctly described by means of one of these indices, although it needs to incorporate second order effects of J_2, which are associated to the centrifugal force derived from the rotating frame used in the computations.

The procedure developed here needs only the integration of accelerations, and, for this reason, is of general application to perturbed Keplerian problems, not being constrained either to J_2 perturbations or to conservative perturbations. Furthermore, it may be extended to the case of perturbed non-Keplerian problems as far as the unperturbed orbit evolves in a closed trajectory. In general, the quadratures required in the computation of the delta-v and corresponding scalar indices will be integrated numerically.

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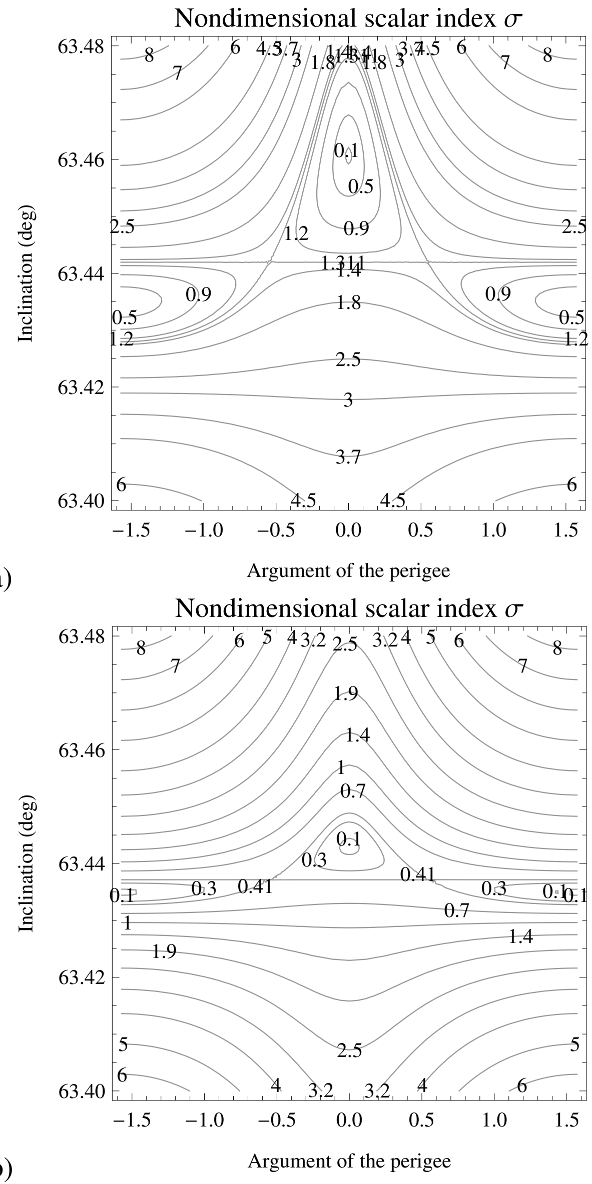

a)

b)

Figure 5: Contours σ, as given by Eq. (33), in the vicinity of the critical inclination. a) $a = \alpha$, $e = 0$. b) $a = \alpha/(1-e)$, $e = 0.6$.

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