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GRAVITATIONAL ACTIONS UPON A TETHER IN A NON-UNIFORM GRAVITY FIELD WITH ARBITRARY NUMBER OF ZONAL HARMONICS

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We develop general closed-form expressions for the mutual gravitational potential, resultant and torque acting upon a rigid tethered system moving in a non-uniform gravity field produced by an attracting body with revolution symmetry, such that an arbitrary number of zonal harmonics is considered. The final expressions are series expansion in two small parameters related to the reference radius of the primary and the length of the tether, respectively, each of which are scaled by the mutual distance between their centers of mass. A few numerical experiments are performed to study the convergence behavior of the final expressions, and conclude that for high precision applications it might be necessary to take into account additional perturbation terms, which come from the mutual Two-Body interaction.

INTRODUCTION

Some problems in celestial mechanics demand an increasing level of accuracy in their modeling. They require the inclusion of small perturbing terms in the equations of motion. One of these often neglected effects is the mutual coupling between the orbital and rotational motion, which in some cases is known to have a substantial impact in the dynamics of extensive bodies, such as planetary satellites,[1] asteroids,[2,3] or large space structures, such as tethers.[4]

An ubiquitous scenario in Astrodynamics is the mutual gravitational interaction between two celestial bodies. To take into account this interaction in detail would be a tremendous problem if we would like to include all possible contributions. Fortunately, the effects of some of these contributions, for example the deformability of the bodies, can be neglected in many situations and simpler formulations can be used.

Regarding these simpler models, an order of complexity can be introduced ranging from the classical Two-Body problem, in which both celestial bodies are modeled as isolated particles, up to the Full Two-Body Problem, where they are considered as rigid bodies. In the latter case, the gravitational potential energy of the system depends on the mass distribution, shape and orientation of each one of the bodies, yielding elaborate right hand sides in the equations of motion. For example, generalization of harmonic expansions for more than one body involves arduous expressions or even untractable for practical applications.

In order to obtain simpler estimates for these mutual interactions, early studies used perturbation theory, assuming that one or both of the bodies are nearly spherical and the coupling between rotational and translational motion is weak and may even be separated from each other.[5] Later studies revisited the problem trying to relax some of these constraints on the bodies, generalizing the gravity field of one or even both bodies, thus leading to the Full Two-Body Problem. Alternatively, recent efforts try to deal with full models and reduce the number of degrees of freedom using canonical simplification theory (see Ref. 6 and references therein).

Later studies revisited the problem trying to relax some of these constraints on the bodies, generalizing the gravity field of one or even both bodies, thus leading to the Full Two-Body Problem. Maciejewski studied systems consisting of a finite number of extended rigid bodies, and proved the existence of relative equilibria.[7] Subsequent research studied the stability of these equilibria and identified fundamental stability criteria.[8,2,3]

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However, even though the general case of bodies with arbitrary shape and mass distribution can be studied *qualitatively* in terms of relative equilibria and their stability, a precise *quantitative* evaluation of the mutual gravitational actions requires in most cases numerical methods, so specific codes are often developed for simulating the roto-translational coupled dynamics.[9]

As a particular case of the Full Two-Body Problem, the motion of a linear body in a spherical gravity field has also been studied for practical applications on tethered space missions.[4] The motion of the center of mass of a tethered system turns out not to be Keplerian. For the case of a massless tether, this can be easily handled by studying the motion of the two end-masses of the tether along with the constraint that their distance must remain constant.

However, when the mass of the tether needs to be considered, this leads to the necessity of treating the tether as an extensive body in order to properly compute the gravity force resultant and torque upon it.[4] Usually, the primary is considered as a spherical body, for simplicity. Considering a non-uniform gravity field of the primary has only been accomplished numerically, or those analytic attempts have just considered a handful of terms of the primary's expansions in spherical harmonics.[10] However, as far as the authors are aware, never before a closed-form analytic formulation has been obtained for the gravitational actions of a tether in a gravity field where an arbitrary number or zonal harmonics of the primary body are retained.

In the current paper, we expose a detailed derivation of the gravitational actions upon an extensive tether orbiting a primary with infinitely many zonal harmonics, as a simplified case of a restricted Full Two-Body Problem. The main part of these derivations was addressed in the Engineering Thesis of Reference 11. The general expressions are given as a double embedded summation in two small parameters, and a brief look is taken at the convergence rates of these series for some test scenarios. Lastly, the effect of the zonal harmonics in the planar equatorial motion of a tether is briefly discussed.

DYNAMICS OF THE TWO-BODY PROBLEM

For a rigid body of total mass m and center of mass G, the equations governing its dynamics with respect to an inertial frame are

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\,\vec{v}_G) = \vec{R}, \qquad \frac{\mathrm{d}\vec{H}_G}{\mathrm{d}t} = \vec{M}_G$$

where $\vec{v}_G$ is the velocity of its center of mass, $\vec{R}$ is the force vector acting upon the body, and $\vec{H}_G$ and $\vec{M}_G$ are its angular momentum vector and external torque at G, respectively. This leads to a system of six degrees of freedom.

When we have two bodies instead, affected solely by their mutual gravitational interaction, the former equations apply to each of the bodies, yielding

$$\begin{aligned} m_1\,\ddot{\vec{r}}_1 &= \vec{R}_{12} \\ m_2\,\ddot{\vec{r}}_2 &= \vec{R}_{21} \\ \bar{\bar{I}}_{G_1} \circ \dot{\vec{\omega}}_1 + \vec{\omega}_1 \wedge \left(\bar{\bar{I}}_{G_1} \circ \vec{\omega}_1\right) &= \vec{M}_{12} \\ \bar{\bar{I}}_{G_2} \circ \dot{\vec{\omega}}_2 + \vec{\omega}_2 \wedge \left(\bar{\bar{I}}_{G_2} \circ \vec{\omega}_2\right) &= \vec{M}_{21} \end{aligned}$$

where $\vec{r}_i$, $\vec{\omega}_i$ and $\bar{\bar{I}}_{G_i}$ are, for $i = 1, 2$, the position vector, angular velocity vector and inertia tensor at the center of mass for each one of the bodies. Additionally, $\vec{R}_{ij}$ and $\vec{M}_{ij}$ are the gravitational force and torque, respectively, that the body i perceives due to the presence of the body j. The kinematic equations relating the attitude of each body with its angular velocity vector $\vec{\omega}_i$ should be added to these equations. The system has, in principle, twelve degrees of freedom. However, as only their mutual gravitational interaction is considered, there are no external forces upon the system, so

$$\vec{R}_{12} + \vec{R}_{21} = \vec{0} \tag{1}$$

$$\vec{M}_{12} + \vec{M}_{21} + \vec{R}_{21} \times (\vec{r}_2 - \vec{r}_1) = \vec{0} \tag{2}$$

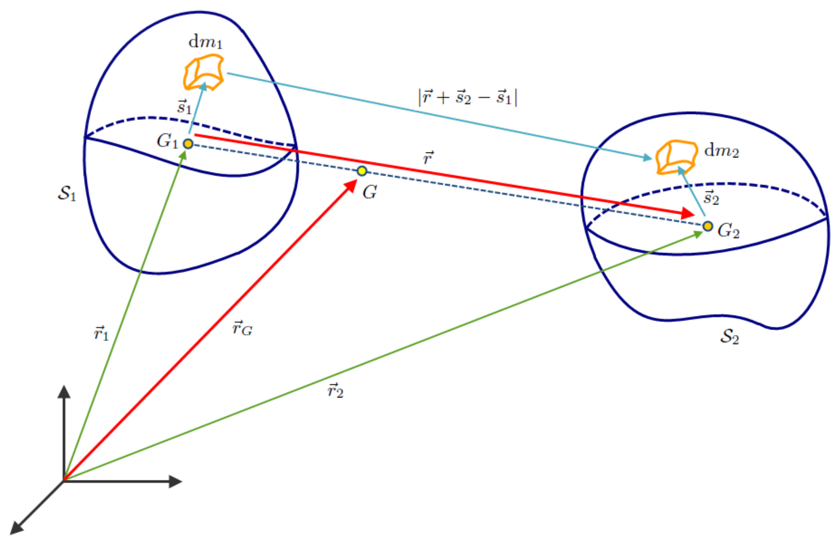


Figure 1: **Geometric layout of the Two-Body Problem and nomenclature**

Consequently, considering equation (1), the translational motion can be more conveniently broken down into 1) the motion of the center of mass, G, which remains unperturbed, and 2) the relative motion of the two bodies around the G. This can be done introducing the generalized coordinates $\vec{r}_G$ and $\vec{r}$ (see Fig. 1), such that

$$\left.\begin{aligned} \vec{r} &= \vec{r}_2 - \vec{r}_1 \\ \vec{r}_G &= \frac{m_1}{m}\,\vec{r}_1 + \frac{m_2}{m}\,\vec{r}_2 \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} \vec{r}_1 &= \vec{r}_G - \frac{m_2}{m}\,\vec{r} \\ \vec{r}_2 &= \vec{r}_G + \frac{m_1}{m}\,\vec{r} \end{aligned}\right.$$

In absence of external forces, $\ddot{\vec{r}}_G = \vec{0}$, so the translational part of the motion reduces to solving the equation

$$\ddot{\vec{r}} = \frac{m}{m_1\,m_2}\,\vec{R}_{21}$$

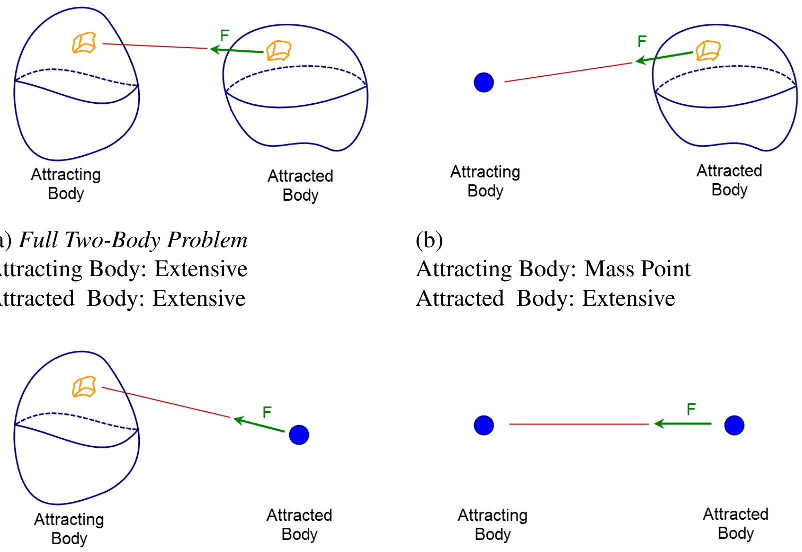
where the main concern is the proper evaluation of the force vector $\vec{R}_{21}$. In the most general case, both bodies are extensive. By this we mean that the mass of the body i is distributed in a finite volume $\mathcal{S}_i$, such that each differential mass element, $\mathrm{d}m_i$, is located with respect to its center of mass by the vector $\vec{s}_i$ (see Fig. 1). Therefore, in order to evaluate the macroscopic gravitational interaction force that the body $i = 1$ exerts upon the body $i = 2$, it is necessary to integrate the Newtonian attraction between every pair of mass elements $\mathrm{d}m_1$ and $\mathrm{d}m_2$, by means of the double volume integral

$$\vec{R}_{21} = -\mathcal{G}\int_{\mathcal{S}_2}\int_{\mathcal{S}_1}\frac{\vec{r}+\vec{s}_2-\vec{s}_1}{|\vec{r}+\vec{s}_2-\vec{s}_1|^3}\,\mathrm{d}m_1\,\mathrm{d}m_2 \tag{3}$$

Analogously, there are no external torques and due to the equation (2) the angular momentum $\vec{H}_G$ remains constant, and the main issue for the rotational dynamics is the adequate quantification of the torque $\vec{M}_{21}$. Evaluating the torque that every mass element $\mathrm{d}m_1$ creates upon every mass elements $\mathrm{d}m_2$ due to their mutual gravitational attraction, and integrating twice to obtain the macroscopic torque, we get

$$\vec{M}_{21} = -\mathcal{G}\int_{\mathcal{S}_2}\int_{\mathcal{S}_1}\frac{\vec{s}_2\wedge\vec{r}-\vec{s}_2\wedge\vec{s}_1}{|\vec{r}+\vec{s}_2-\vec{s}_1|^3}\,\mathrm{d}m_1\,\mathrm{d}m_2 \tag{4}$$

When we study the motion of these two bodies using Equations (3) and (4), we refer to this as the *Full* Two-Body Problem, schematically represented in Figure 2a. However, evaluating these integrals can be fairly



(a) *Full Two-Body Problem*
Attracting Body: Extensive
Attracted Body: Extensive

(b)
Attracting Body: Mass Point
Attracted Body: Extensive

(c)
Attracting Body: Extensive
Attracted Body: Mass Point

(d)
Attracting Body: Mass Point
Attracted Body: Mass Point

Figure 2: **Conceptual drawings illustrating different approaches to the Two-Body Problem**

difficult in a general case, to the extent that a numerical computation might be the only feasible approach. Therefore, simplifying assumptions are usually made. Different assumptions are more or less adequate for a certain aim, so it is one's decision to choose which simplifications suffice, knowing that the more complex approaches better represent the physics, but they are also more arduous to work with.

For example, if one considers both bodies are sufficiently far from each other, then $|\vec{s}_1| \ll |\vec{r}|$ and $|\vec{s}_2| \ll |\vec{r}|$, so the denominators in integrals (3) and (4) can be approximated by series expansions in powers of $\vec{s}_1/|\vec{r}|$ and $\vec{s}_2/|\vec{r}|$, up to the desired degree of approximation.

One can go a step further and take the limiting case where one of the bodies has a null volume, and thus all the mass is reduced to a point, i.e. $|\vec{s}_i| \to 0$ and a volume integral vanishes. This simplification can be seen from two different perspectives. If one of the two bodies is so much more massive than the other, as for example $m_1 \gg m_2$, then one can assume a *Restricted* Two-Body Problem, were basically the body m_2 orbits around m_1, while m_1 does not notice at all the gravitational pull of m_2. Hence, we can refer to m_1 as the *attracting* body, and to m_2 as the *attracted* body. Hence the two obvious possibilities are considering m_1 or m_2 as an extensive body, and assume the other as a mass point.

For instance, if one reduces the attracting body to a mass point ($|\vec{s}_1| \to 0$) while considering the orbiting body as extensive (see Figure 2b), this model can be used to study the orbital attitude dynamics of a satellite around a spherical planet. The attracting mass point would cause a slightly different gravitational pull upon every mass element $\mathrm{d}m_2$ of the extensive body, thus requiring a single volume integration in $\mathcal{S}_2$, and allowing us to compute a gravitational torque upon the orbiter. In this case, Eqs. (3) and (4) reduce to

$$\vec{R}_{21} = -\mathcal{G}\, m_1 \int_{\mathcal{S}_2} \frac{\vec{r}+\vec{s}_2}{|\vec{r}+\vec{s}_2|^3}\, \mathrm{d}m_2$$
$$\vec{M}_{21} = -\mathcal{G}\, m_1 \int_{\mathcal{S}_2} \frac{\vec{s}_2 \wedge \vec{r}}{|\vec{r}+\vec{s}_2|^3}\, \mathrm{d}m_2$$

where for a small orbiter $|\vec{s}_2| \ll |\vec{r}|$ seems a fair assumption, and thus the former equations can be further simplified. Hence, expanding the former expressions in powers of s_2/r and retaining up to second order terms, we get

$$\vec{R}_{21} \simeq -\frac{\mathcal{G}\, m_1\, m_2}{r^3}\vec{r} - 3\,\frac{\mathcal{G}\, m_1\, m_2}{r^5}\left[\left(I_{G_2} - \frac{5}{2\,r^2}\,\vec{r}\circ\bar{\bar{I}}_{G_2}\circ\vec{r}\right)\vec{r} + \bar{\bar{I}}_{G_2}\circ\vec{r}\right]$$
$$\vec{M}_{21} \simeq -3\,\frac{\mathcal{G}\, m_1\, m_2}{r^5}\,\vec{r}\wedge\left(\bar{\bar{I}}_{G_2}\circ\vec{r}\right)$$

where $\bar{\bar{I}}_{G_2}$ is the tensor of inertia of the attracted body at its center of mass, G_2.

The opposite situation is encountered when we consider instead the orbiting body as a mass point ($|\vec{s}_2| \to 0$), and the attracting body as extensive (see Fig. 2c), thus assuming that every $\mathrm{d}m_1$ of the attracting body is causing a different attracting force upon the mass point orbiter, which also implies a single volume integral. A typical application example is the precise orbit propagation of a satellite orbiting a planet with a non-uniform gravity field which perturbs the trajectory. In this case, Eqs. (3) and (4) reduce to

$$\vec{R}_{21} = -\mathcal{G}\, m_2 \int_{\mathcal{S}_1} \frac{\vec{r}-\vec{s}_1}{|\vec{r}-\vec{s}_1|^3}\,\mathrm{d}m_1$$
$$\vec{M}_{21} = \vec{0}$$

Finally, the simplest model would be to consider both bodies as mass points attracting each other. This provides a computationally fast solution, though it is not necessarily accurate for close objects. However, the further both bodies are from each other, this approximation becomes increasingly precise. This is clearly the case of the Solar System planets, which are millions of kilometers away from each other, so this approach is often used in evaluating the third body perturbations due to the presence of distant celestial bodies, such as the lunisolar perturbation on an Earth orbiting satellite. Figure 2d represents this model, and gravitational actions reduce to their simplest expression

$$\vec{R}_{21} = -\mathcal{G}\,\frac{m_1\, m_2}{r^3}\vec{r}$$
$$\vec{M}_{21} = \vec{0}$$

So, in the most general approach, namely the Full Two-Body Problem, a double volume integration is required, which usually makes this approach difficult and computationally expensive.

However, when one of the two bodies is so slim as a tether, this body would very well admit to be treated as a linear body, and one of the volume integrals would turn into a linear integral, greatly simplifying the resulting equations, *so in the following we shall consider the Full Two-Body approach where the attracting celestial body is extensive, as well as the tether itself, in order to evaluate gravity forces and torques upon the tether*.

THE TETHERED SYSTEM

The *tethered system* under consideration, or by extension simply the *tether*, is made up of two spacecrafts tied one to another by a tether, considered as a long, fixed-length, rigid rod. This model is commonly referred to as a *dumbbell model* in case of a massless tether, or *extended* dumbbell model when the mass of the tether is taken into consideration. We shall take the latter approach in the ongoing derivations. Therefore, both spacecrafts will be modeled as mass points of masses m_1 and m_2 respectively, whereas the tether, of constant length L_T and mass m_T, will be considered to have a linear* mass density, ρ_L, where $\rho_L = m_T/L_T$ if the density is homogeneous. The center of mass of the tethered system, G, is then located on the tether itself, somewhere in-between both end-masses, at a distance L_1 from the mass m_1 and a distance L_2 from the mass m_2, such that $L_T = L_1 + L_2$, as shown in Figure 3.

*Space tethers are so long and thin that considering them as one-dimensional elements proves a fair assumption.

The mass geometry of a tether is then fully defined by the parameters m_1, m_2 and m_T. However, we shall conveniently introduce three equivalent parameters, namely m, Λ and ϕ defined by

$$m = m_1 + m_2 + m_T \tag{5}$$
$$\Lambda = \frac{m_T}{m} \tag{6}$$
$$\nu_1 = \cos^2\phi = \frac{1}{m}\left(m_1 + \frac{1}{2}\,m_T\right) \tag{7}$$
$$\nu_2 = \sin^2\phi = \frac{1}{m}\left(m_2 + \frac{1}{2}\,m_T\right) \tag{8}$$

where m is the total mass of the tethered system, $\Lambda \in [0,1]$ is the non-dimensional mass of the tether (frequently $\Lambda \ll 1$), ν_1 and ν_2 are the reduced end-masses, and $\phi \in [0, \pi/2]$ is known as the mass angle, since the reduced masses satisfy $\nu_1 + \nu_2 = 1$.

The inverse transformations can be accomplished with the relations

$$m_1 = m\left(\cos^2\phi - \frac{1}{2}\,\Lambda\right)$$
$$m_2 = m\left(\sin^2\phi - \frac{1}{2}\,\Lambda\right)$$
$$m_T = m\,\Lambda.$$

If we place the origin of the coordinate system O at the center of mass of the attracting body, we can define the position vector of the tether center of mass, $\vec{r}_G$, as

$$\vec{r}_G = r_G \cdot \vec{u}_G, \qquad r_G = \mid \vec{r}_G \mid$$

Let s be an arclength coordinate indicating the linear position of any tether mass element, $\mathrm{d}m$, measured from the point G along the tether. The direction of the tether is given by the unity vector $\vec{u}$, arbitrarily pointing from the end-mass m_1 to m_2. This allows us to define the angle α as the one formed between the unit vectors $\vec{u}_G$ and $\vec{u}$ (see Fig. 3).

By means of these unit vectors, we can express the position of any $\mathrm{d}m$ of the tether, $\vec{r}$, as

$$\vec{r} = r_G \cdot \vec{u}_G + s \cdot \vec{u}$$

or more conveniently expressed in terms of the new non-dimensional variable $\eta = s/r_G$, as

$$\vec{r} = r_G(\vec{u}_G + \eta \cdot \vec{u}) \tag{9}$$

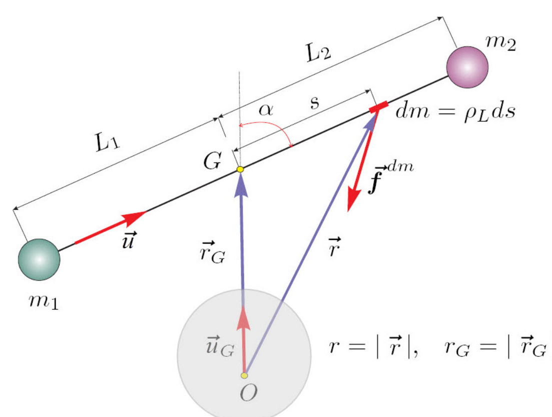


Figure 3: **Scheme showing the rigid tethered system considered in the analysis and the nomenclature used for geometrical variables and vectors in the description of the extended dumbbell model**

It is of great interest to use the variable η instead of s, since for practical cases $\eta \ll 1$. Note that the modulus of the vector $\vec{r}$ can be expressed as

$$r = |\vec{r}| = r_G\sqrt{1 + 2\,\eta\,\cos\alpha + \eta^2} \tag{10}$$

and therefore we will be able to expand its inverse in Legendre series; note that a Legendre series is a particular case of ultraspherical series,[12] that we briefly describe in appendix . Thus, to expand $1/r$ in

Legendre series we may simply use Eq. (30) with $\lambda = 1/2$ and $\gamma = -\cos\alpha$, along with the relation (31), which yields

$$\frac{1}{r} = \frac{1}{r_G} \sum_{n=0}^{\infty} (-1)^n \, \eta^n \, P_n(\cos\alpha) \tag{11}$$

where P_n is the n^{th} degree Legendre Polynomial.

Notice that the gravitational attraction vector upon every single $\mathrm{d}m$ of the tether, $\vec{f}^{dm}$, is not necessarily aligned with the position vector $\vec{r}$ in general, since the gravity field produced by the extensive body located at O is not uniform.

GRAVITATIONAL ACTIONS IN A SPHERICAL GRAVITY FIELD

If the gravity field of the attracting body is considered uniform or spherical, then it is by all means equivalent to a mass point of mass M, and hence the Full Two-Body Problem reduces to the approach sketched in Figure 2b, where the attracting body is modeled as a mass point and the tether as an extensive body.

Gravitational Potential

Under these hypothesis, the mutual gravitational potential, V_s, is given by

$$V_s = -\int_{\mathcal{S}} \frac{\mu}{r} \, \mathrm{d}m,$$

where $\mu = \mathcal{G}M$ is the gravitational parameter of the primary, r is the modulus of the position vector of each tether element $\mathrm{d}m$, and the integral above extends to the whole volume $\mathcal{S}$ occupied by the tethered system. Using Eq. (10) the integral takes the form

$$V_s = -\frac{\mu}{r_G} \int_{\mathcal{S}} \frac{\mathrm{d}m}{\sqrt{1 + 2\,\eta\,\cos\alpha + \eta^2}}$$

and after expanding the denominator in Legendre series, as in Eq. (11), the gravitational potential takes the form

$$V_s = -\frac{\mu}{r_G} \sum_{n=0}^{\infty} (-1)^n P_n(\cos\alpha) \, I_n \tag{12}$$

with the functions I_n given by

$$I_n = \int_{\mathcal{S}} \eta^n \, \mathrm{d}m. \tag{13}$$

These integral functions, I_n, are nothing more but the n-th order inertia moments, and can be calculated for a tethered system, due to its particular homogeneous linear mass distribution, in a simple manner up to the desired n-th order. If we apply the change of variable

$$\mathrm{d}m = \rho_L \, \mathrm{d}s = \rho_L \, r_G \, \mathrm{d}\eta$$

and expand the integral, we get

$$I_n = m_1 \, \eta_1^n + m_2 \, \eta_2^n + \rho_L \, r_G \int_{\eta_1}^{\eta_2} \eta^n \, \mathrm{d}\eta$$

where the integration boundaries are related to the geometric and mass properties of the tether through

$$\eta_1 = -\frac{L_1}{r_G} = -\frac{L_T}{r_G} \cdot \sin^2\phi$$
$$\eta_2 = +\frac{L_2}{r_G} = +\frac{L_T}{r_G} \cdot \cos^2\phi.$$

It is also preferable to express these higher order inertia moments, I_n, not as a function of the variables $\{m_1, m_2, \rho_L\}$, but rather as a function of $\{m, \phi, \Lambda\}$ instead, using Eqs. (5) to (8). Hence, solving the integrals and with adequate trigonometric manipulations, we easily obtain the I_n functions as follows

$$\begin{aligned}
I_0 &= m \\
I_1 &= 0 \\
I_2 &= \frac{m}{24}\left(\frac{L_T}{r_G}\right)^2 \Big(3 - 3\,\cos(4\phi) - 4\,\Lambda\Big) \\
I_3 &= \frac{m}{4}\left(\frac{L_T}{r_G}\right)^3 \cos(2\phi)\,\Big(\sin^2(2\phi) - \Lambda\Big) \\
I_4 &= \frac{m}{640}\left(\frac{L_T}{r_G}\right)^4 \Big(-30\,\cos^2(4\phi) - 20\,\cos(4\phi) - 80\,\Lambda\,\cos(4\phi) + 50 - 112\,\Lambda\Big) \\
I_5 &= \frac{m}{24}\left(\frac{L_T}{r_G}\right)^5 \Big(3 - 3\,\cos^4(2\phi) - 5\,\Lambda\,\cos^2(2\phi) - 3\,\Lambda\Big) \\
I_6 &= \ldots
\end{aligned} \tag{14}$$

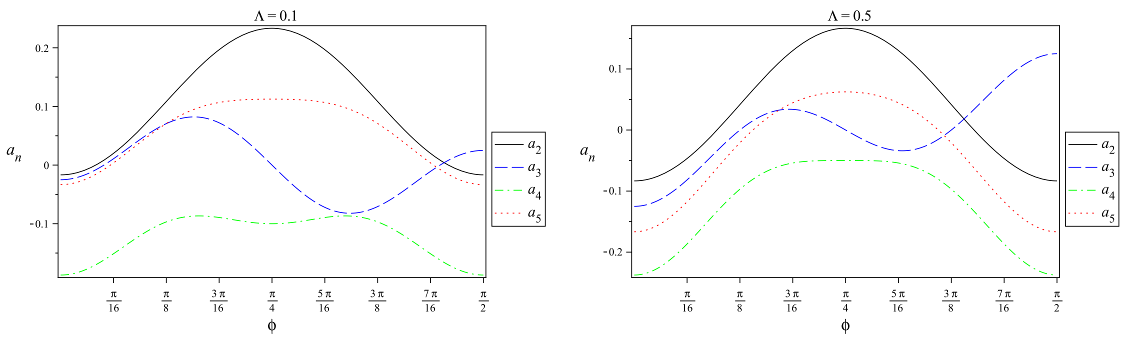


Figure 4: **Functions $a_n(\phi, \Lambda)$ up to $n = 5$, for two different values of parameter Λ**

At this point, we realize that the functions I_n obey the form

$$I_n = m\left(\frac{L_T}{r_G}\right)^n a_n(\phi, \Lambda) \tag{15}$$

where the nondimensional coefficients $a_n(\phi, \Lambda)$ do not depend on the size of the tether, but just its mass distribution, and it should be noted that $a_n \geqslant 0$ (see Fig. 4). This notation gives a more clarifying idea of how negligible each term is, since $\left(\frac{L_T}{r_G}\right) \ll 1$. Thus, by identifying terms between the calculated I_n and their proposed form, the coefficients $a_n(\phi, \Lambda)$ may be easily obtained:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= \tfrac{1}{24}\Big(3 - 3\,\cos(4\phi) - 4\,\Lambda\Big) \\
a_3 &= \tfrac{1}{4}\,\cos(2\phi)\,\Big(\sin^2(2\phi) - \Lambda\Big) \\
a_4 &= \tfrac{1}{640}\Big(-30\,\cos^2(4\phi) - 20\,\cos(4\phi) - 80\,\Lambda\,\cos(4\phi) + 50 - 112\,\Lambda\Big) \\
a_5 &= \tfrac{1}{24}\Big(3 - 3\,\cos^4(2\phi) - 5\,\Lambda\,\cos^2(2\phi) - 3\,\Lambda\Big) \\
a_6 &= \ldots
\end{aligned} \tag{16}$$

Note that for a non-homogeneous mass distribution (as for a varying cross-section tether), the inertia moments I_n would not be those collected in Eqs. (14), but they would instead need to be calculated accordingly, from its definition in Eq. (13).

For a massive tether the mass angle ϕ ranges in the interval $[\phi_{min}, \phi_{max}]$ whose end points are given by

$$m_2 = 0 \quad \Rightarrow \quad \phi_{min} = \arcsin(\sqrt{\frac{\Lambda}{2}}), \tag{17}$$

$$m_1 = 0 \quad \Rightarrow \quad \phi_{max} = \arccos(\sqrt{\frac{\Lambda}{2}}), \tag{18}$$

They only depend on Λ and fulfill the relation

$$\phi_{min} + \phi_{max} = \frac{\pi}{2}$$

Since Λ is usually small, it is possible to use the approximate expressions:

$$\phi_{min} \approx 0, \qquad \phi_{max} \approx \frac{\pi}{2}$$

This is the reason why, in figure 4, ϕ ranges in the interval $[0, \frac{\pi}{2}]$.

Gravitational Resultant

The gravitational resultant, $\vec{R}_s$, is given by the integral

$$\vec{R}_s = -\int_{\mathcal{S}} \frac{\mu}{r^2}\ \vec{u}_r \,\mathrm{d}m$$

that extends to the whole volume $\mathcal{S}$ occupied by the tether. The unit vector $\vec{u}_r$ can more conveniently be expressed as a function of the known unit vectors $\vec{u}_G$ and $\vec{u}$, and the variable η, as

$$\vec{u}_r = \frac{\vec{r}}{r} = \frac{\vec{u}_G + \eta\,\vec{u}}{\sqrt{1 + 2\,\eta\,\cos\alpha + \eta^2}} \tag{19}$$

so that the integral takes the form

$$\vec{R}_s = -\frac{\mu}{r_G^2} \int_{\mathcal{S}} \frac{\vec{u}_G + \eta\,\vec{u}}{(1 + 2\,\eta\,\cos\alpha + \eta^2)^{\frac{3}{2}}}\,\mathrm{d}m$$

Operating similarly to the previous section, the denominator can be expanded in series of ultraspherical polynomials, which are a generalization of Legendre polynomials.[12] We have considered appropriate to include an appendix introducing ultraspherical polynomials and some of their properties that will be used later in this paper. Thus, making use of the relation (30) to expand the denominator and identifying the I_n integral functions, the gravitational resultant becomes

$$\vec{R}_s = -\frac{\mu}{r_G^2} \sum_{n=0}^{\infty} (-1)^n\, C_n^{\left(\frac{3}{2}\right)}(\cos\alpha)\ \left(I_n\,\vec{u}_G + I_{n+1}\,\vec{u}\,\right). \tag{20}$$

where the function $C_n^{\left(\frac{3}{2}\right)}(x)$ is the corresponding *Ultraspherical Polynomial*, also known as *Gegenbauer Polynomial*.

Gravitational Torque

In order to obtain the gravitational torque acting upon the center of mass of the tethered system, $\vec{M}_s$, we must integrate in the whole volume of the tether, $\mathcal{S}$, the gravitational force acting upon every single $\mathrm{d}m$ of the tether considering its distance to the center of mass. In other words, the torque is calculated as

$$\vec{M}_s = -\int_{\mathcal{S}} s\,\vec{u} \wedge \frac{\mu}{r^2}\vec{u}_r\,\mathrm{d}m.$$

Taking into account that $s = r_G\,\eta$, substituting $\vec{u}_r$ and expanding in ultraspherical polynomials as we did for the resultant $\vec{R}_s$, the gravitational torque finally takes the form

$$\vec{M}_s = -\frac{\mu}{r_G} \sum_{n=0}^{\infty} (-1)^n\, C_n^{\left(\frac{3}{2}\right)}(\cos\alpha)\, I_{n+1}\,\vec{u} \wedge \vec{u}_G \tag{21}$$

GRAVITATIONAL ACTIONS IN A NON-UNIFORM GRAVITY FIELD

According to the Full Two-Body formulation presented in §, the mutual gravitational potential is defined as

$$V = \int_{\mathcal{S}'} \int_{\mathcal{S}} V^{\,\mathrm{d}m\,\mathrm{d}M}$$

where $V^{\,\mathrm{d}m\,\mathrm{d}M}$ is the differential gravitational potential between a tether mass element, $\mathrm{d}m$, and a mass element of the primary, $\mathrm{d}M$, and must be integrated both over the volume $\mathcal{S}$ occupied by the tether and the volume $\mathcal{S}'$ occupied by the primary. If we integrate over a non-homogeneous primary body, $\mathcal{S}'$, the resulting differential potential, $V^{\,\mathrm{d}m}$, can be approximated by the well-known solid spherical harmonics expansion

$$V^{\,\mathrm{d}m} = -\frac{\mu}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R}{r}\right)^{l} P_{lm}(\sin\varphi) \Big[C_{lm} \cdot \cos(m\,\theta) + S_{lm} \cdot \sin(m\,\theta) \Big] \,\mathrm{d}m$$

where C_{lm} and S_{lm} are the potential coefficients, also known as *Stokes* coefficients, μ is the gravitational parameter of the primary or attracting body and R is its reference radius. The P_{lm} are the *associated Legendre functions*, also known as *Ferrer's functions*, and angles φ, θ are respectively the latitude and longitude in a equatorial body-fixed reference frame. Notice that this expansion is convergent just as long as $r > R$.

The expression for the gravity potential might be interestingly simplified in the particular case in which the considered celestial body has a revolution symmetry. It happens that this symmetry configuration makes all potential coefficients vanish, except for the C_{l0} subset. These are known as *zonal* coefficients, as they do not depend on the longitude, and are usually redefined as $J_l = -C_{l0}$. In terms of only the zonal spherical harmonics, the gravity field specific potential reduces to

$$V^{\,\mathrm{d}m} = -\frac{\mu}{r} \left[1 - \sum_{l=2}^{\infty} \bar{J}_l \left(\frac{R}{r}\right)^{l} \bar{P}_n(\sin\varphi) \right] \mathrm{d}m \tag{22}$$

Note that Eq. (22) has two terms, the first of which corresponds to the uniform or spherical gravity field ($V^{\,\mathrm{d}m} = -\mu/r\,\mathrm{d}m$), and the seconds contains the contributions of the zonal harmonics.

Even though retaining just zonal harmonics and omitting tesseral and sectorial harmonics may seem like a large simplification, results are still highly valuable, because 1) the highest perturbing harmonic is the J_2, and 2) in studying the long term dynamics, the effects of tesseral and sectorial harmonics average out and only the effect of the J_l terms remains.

Now, we shall recalculate the gravitational potential, resultant and torque upon a tether in a non-uniform gravity field, including the perturbations due to the zonal harmonics, J_l. Hence, we are going to deal with a Full Two-Body Problem, as in Figure 2a, where both the tether and primary body are considered as extensive bodies, with only two simplifying assumptions, 1) the tether is considered as a linear one-dimensional element, and 2) the attracting body is assumed to have revolution symmetry, so that only zonal harmonics are retained.

Gravitational Potential

Under these hypothesis, the mutual gravitational potential, V, is given by

$$V = \int_{\mathcal{S}} V^{\,\mathrm{d}m} = -\int_{\mathcal{S}} \frac{\mu}{r} \left[1 - \sum_{l=2}^{\infty} J_l \left(\frac{R}{r}\right)^{l} P_l(\sin\varphi) \right] \mathrm{d}m$$

where the integral extends to the whole volume $\mathcal{S}$ occupied by the tethered system. This potential can be broken down into a summation of two terms,

$$V = V_s + V_j,$$

such that the first term, V_s, turns out to be the potential for the case of a spherical attracting body, whose solution we obtained earlier in Eq. (12), and the second term,

$$V_j = \int_{\mathcal{S}} \frac{\mu}{r} \sum_{l=2}^{\infty} J_l \left(\frac{R}{r}\right)^l P_l(\sin\varphi)\, \mathrm{d}m$$

contains the contribution of the zonal harmonics to the whole potential V. Using Eq. (10) the potential takes the form

$$V_j = \frac{\mu}{r_G} \sum_{l=2}^{\infty} J_l \left(\frac{R}{r_G}\right)^l \int_{\mathcal{S}} \frac{P_l(\sin\varphi)}{(1+2\,\eta\,\cos\alpha+\eta^2)^{\frac{l+1}{2}}}\, \mathrm{d}m.$$

Now, it is convenient to develop the Legendre functions, $P_l(\sin\varphi)$, as a summation of monomials using the relation (33), yielding

$$V_j = \frac{\mu}{r_G} \sum_{l=2}^{\infty} J_l \left(\frac{R}{r_G}\right)^l \int_{\mathcal{S}} \frac{\sum_{p=0}^{l} b_{lp}(1/2)\cdot \sin^p\varphi}{(1+2\,\eta\,\cos\alpha+\eta^2)^{\frac{l+1}{2}}}\, \mathrm{d}m.$$

The variable φ stands for the latitude of each tether element $\mathrm{d}m$ with respect to the equatorial plane of the primary, and can be related to the latitude of the tether center of mass, φ_G, by means of the geometric relation

$$\sin(\varphi) = \frac{\sin(\varphi_G) + \eta\,(\vec{u}\cdot\vec{k})}{\sqrt{1+2\,\eta\,\cos\alpha+\eta^2}} \tag{23}$$

where the unit vector $\vec{k}$ is aligned with the north pole of the primary, i.e. perpendicular to the equatorial plane towards increasing value of the latitude. The use of the variable φ_G instead is advisable, since is it known from the tether's position vector $\vec{r}_G$, whereas the latitude φ depends on the linear position η along the tether, and Eq. (23) makes this dependence explicit. Thus, substituting the value of $\sin(\varphi)$ in the gravitational potential V_j, we arrive at

$$V_j = \frac{\mu}{r_G} \sum_{l=2}^{\infty} \sum_{p=0}^{l} J_l \left(\frac{R}{r_G}\right)^l b_{lp}(1/2) \int_{\mathcal{S}} \frac{\left(\sin(\varphi_G)+\eta\,(\vec{u}\cdot\vec{k})\right)^p}{(1+2\,\eta\cos\alpha+\eta^2)^{\frac{l+p+1}{2}}}\, \mathrm{d}m$$

At this point, recalling the Newton binomial permits the expansion

$$\left(\sin(\varphi_G)+\eta\,(\vec{u}_1\cdot\vec{k}_4)\right)^p = \sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} \sin^{p-q}(\varphi_G)\,(\vec{u}\cdot\vec{k})^q\,\eta^q$$

that leads to the potential to be written as

$$\begin{aligned} V_j =& \frac{\mu}{r_G} \sum_{l=2}^{\infty} \sum_{p=0}^{l} \sum_{q=0}^{p} J_l \left(\frac{R}{r_G}\right)^l b_{lp}(1/2) \begin{pmatrix} p \\ q \end{pmatrix} \sin^{p-q}(\varphi_G)\,(\vec{u}\cdot\vec{k})^q \cdot \\ &\cdot \int_{\mathcal{S}} \frac{\eta^q}{(1+2\,\eta\cos\alpha+\eta^2)^{\frac{l+p+1}{2}}}\, \mathrm{d}m. \end{aligned}$$

Now, taking a look at the denominator inside the integral, we realize that it is expandable in a series of ultraspherical polynomials, according to the relation (30) and using the property (31). Also, making use of the integral functions I_n introduced earlier (Eqs. (13) & (14)), the contribution of the zonal harmonics to the gravitational potential finally takes the form

$$\begin{aligned} V_j =& \frac{\mu}{r_G} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l} \sum_{q=0}^{p} (-1)^n J_l \left(\frac{R}{r_G}\right)^l b_{lp}(1/2) \begin{pmatrix} p \\ q \end{pmatrix} \sin^{p-q}(\varphi_G) \cdot \\ &\cdot (\vec{u}\cdot\vec{k})^q\, C_n^{\left(\frac{l+p+1}{2}\right)}(\cos\alpha)\, I_{n+q}, \end{aligned}$$

or expressed in terms of the coefficients $a_n(\phi, \Lambda)$ instead, as indicated by Eqs. (15) & (16), we arrive at

$$V_j = \frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l} \sum_{q=0}^{p} (-1)^n \, J_l \left(\frac{R}{r_G}\right)^l b_{lp}(1/2) \begin{pmatrix} p \\ q \end{pmatrix} \sin^{p-q}(\varphi_G) \cdot$$
$$\cdot \, (\vec{u} \cdot \vec{k})^q \, C_n^{\left(\frac{l+p+1}{2}\right)}(\cos\alpha) \, a_{n+q} \left(\frac{L_T}{r_G}\right)^{n+q} . \tag{24}$$

The complete mutual gravitational potential, V, is the sum of V_s and V_j.

Gravitational Resultant

Knowing the differential gravitational potential generated by a primary with revolution symmetry, $V^{\,\mathrm{d}m}$, we get the gravitational resultant upon a tether by integrating $-\nabla V^{\,\mathrm{d}m}$ over each $\mathrm{d}m$ of the tether, i.e.

$$\vec{R} = \int_{\mathcal{S}} \nabla \left\{ \frac{\mu}{r} \Big[1 - \sum_{l=2}^{\infty} J_l \left(\frac{R}{r}\right)^l P_l(\sin\varphi) \Big] \right\} \mathrm{d}m$$

The gradient operator in spherical coordinates is given by

$$\nabla = \frac{\partial}{\partial r} \vec{u}_r + \frac{1}{r} \frac{\partial}{\partial \varphi} \vec{u}_\varphi + \frac{1}{r\cos\varphi} \frac{\partial}{\partial \theta} \vec{u}_\theta,$$

though the revolution symmetry simplifies the operator, since $\frac{\partial}{\partial\theta} = 0$. Again, the above integral expression for the gravity resultant may be broken down into a summation of two terms,

$$\vec{R}(r, \varphi) = \vec{R}_s(r) + \vec{R}_j(r, \varphi),$$

such that the first term, $\vec{R}_s$, depends only on the distante r but not on the latitude φ, and turns out to be equal to the resultant for the case of a spherical attracting body, whose solution we obtained earlier in Eq. (20), and the second term,

$$\vec{R}_j = - \int_{\mathcal{S}} \nabla \left\{ \frac{\mu}{r} \sum_{l=2}^{\infty} J_l \left(\frac{R}{r}\right)^l P_l(\sin\varphi) \right\} \mathrm{d}m,$$

which does depend on both r and φ, contains the contribution of the zonal harmonics to the complete resultant $\vec{R}$. Applying the gradient operator yields

$$\vec{R}_j = \int_{\mathcal{S}} \frac{\mu}{r^2} \sum_{l=2}^{\infty} J_l \left(\frac{R}{r}\right)^l \left[(l+1)\, P_l(\sin\varphi)\, \vec{u}_r - \frac{\mathrm{d}\, P_l(\sin\varphi)}{\mathrm{d}(\sin\varphi)} \cos\varphi\, \vec{u}_\varphi \right] \mathrm{d}m.$$

The unit vector $\vec{u}_r$ can be expressed in terms of the known vectors $\vec{u}_G$ and $\vec{u}$, and the variable η using the relation (19). Similarly, the unit vector $\vec{u}_\varphi$ can be written as

$$\vec{u}_\varphi = \vec{u}_r \wedge \vec{u}_\theta = \vec{u}_r \wedge \left(\frac{\vec{k} \wedge \vec{u}_r}{|\, \vec{k} \wedge \vec{u}_r \,|} \right) = \frac{\vec{u}_r \wedge (\vec{k} \wedge \vec{u}_r)}{\cos\varphi} = \frac{\vec{k} - \sin(\varphi)\, \vec{u}_r}{\cos\varphi},$$

which after replacing $\vec{u}_r$ leads to

$$\vec{u}_\varphi = \frac{1}{\cos\varphi} \left(\vec{k} - \sin\varphi \, \frac{\vec{u}_G + \eta\, \vec{u}}{\sqrt{1 + 2\,\eta\, \cos\alpha + \eta^2}} \right). \tag{25}$$

Additionally, by the property (32) of ultraspherical polynomials we can compute the differentiation of the Legendre functions $P_l(\sin\varphi)$ as

$$\frac{\mathrm{d}\, P_l(\sin\varphi)}{\mathrm{d}\sin\varphi} = C_{l-1}^{\left(\frac{3}{2}\right)}(\sin\varphi). \tag{26}$$

Now, substituting unit vector $\vec{u}_\varphi$ in the gravitational resultant $\vec{R}_j$ and using the relations (10) and (26), the resultant becomes

$$\begin{aligned}\vec{R}j =& \frac{\mu}{r_G^2} \sum_{l=2}^{\infty} J_l \left(\frac{R}{r_G}\right)^l \cdot \\ & \cdot \int_{S_2} \frac{(l+1)\, P_l(\sin\varphi)\, \vec{u}_r + C_{l-1}^{\left(\frac{3}{2}\right)}(\sin\varphi)\, \sin\varphi\, \vec{u}_r - C_{l-1}^{\left(\frac{3}{2}\right)}(\sin\varphi)\, \vec{k}}{(1 + 2\,\eta\, \cos\alpha + \eta^2)^{\frac{l+2}{2}}}\, \mathrm{d}m.\end{aligned}$$

The integral part above can be broken down in another three integrals and solved separately proceeding analogously as we did for the gravitational potential, i.e. replacing the unit vector $\vec{u}_r$ with Eq. (19), expressing ultraspherical polynomials as summations of monomials according to Eq. (33), replacing $\sin(\varphi)$ with Eq. (23), using the Newton binomials, expanding the denominators into new ultraspherical polynomials and using the functions $a_n(\phi, \Lambda)$. After many mathematical manipulations, the gravitational resultant $\vec{R}_j$ finally takes the form

$$\begin{aligned}\vec{R}_j \;=\; & -\frac{\mu m}{r_G^2} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^{p} (-1)^n\, J_l \left(\frac{L_T}{r_G}\right)^{n+q} \left(\frac{R}{r_G}\right)^l b_{l-1,p}(3/2) \begin{pmatrix} p \\ q \end{pmatrix} \cdot \\ & \cdot \sin^{p-q}(\varphi_G)\; (\vec{u}\cdot\vec{k})^q\, C_n^{\left(\frac{l+p+2}{2}\right)}(\cos\alpha)\, a_{n+q} \vec{k} + \\ & + \frac{\mu m}{r_G^2} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l} \sum_{q=0}^{p} (-1)^n\, J_l \left(\frac{L_T}{r_G}\right)^{n+q} \left(\frac{R}{r_G}\right)^l b_{l,p}(1/2) \begin{pmatrix} p \\ q \end{pmatrix} \cdot \\ & \cdot \sin^{p-q}(\varphi_G)\; (\vec{u}\cdot\vec{k})^q\, (l+1)\, C_n^{\left(\frac{l+p+3}{2}\right)}(\cos\alpha) \left(a_{n+q}\, \vec{u}_G + \left(\frac{L_T}{r_G}\right) a_{n+q+1}\, \vec{u}\right) + \\ & + \frac{\mu m}{r_G^2} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^{p+1} (-1)^n\, J_l \left(\frac{L_T}{r_G}\right)^{n+q} \left(\frac{R}{r_G}\right)^l b_{l-1,p}(3/2) \begin{pmatrix} p+1 \\ q \end{pmatrix} \cdot \\ & \cdot \sin^{p-q+1}(\varphi_G)\; (\vec{u}\cdot\vec{k})^q\, C_n^{\left(\frac{l+p+4}{2}\right)}(\cos\alpha) \left(a_{n+q}\, \vec{u}_G + \left(\frac{L_T}{r_G}\right) a_{n+q+1}\, \vec{u}\right)\end{aligned} \tag{27}$$

The complete gravitational resultant upon the tether, $\vec{R}$, is the vector sum of its contributions, $\vec{R}_s$ and $\vec{R}_j$.

Gravitational Torque

Now, including the zonal harmonics, the gravitational torque, $\vec{M}$, is calculated with the integral

$$\vec{M} = \int_{\mathcal{S}} s\, \vec{u} \wedge \left[\nabla\left(\frac{\mu}{r}\left(1 - \sum_{l=2}^{\infty} J_l \left(\frac{R}{r}\right)^l P_l(\sin\varphi)\right)\right)\right] \mathrm{d}m,$$

which can be, as we did in previous sections, broken down as a sum of two term,

$$\vec{M} = \vec{M}_s + \vec{M}_j,$$

the first of which, $\vec{M}_s$, is equal to the torque in the case of a spherical attracting body and we already calculated it in Eq. (21). The other term, $\vec{M}_j$, contains the contribution of the J_l zonal harmonics of the primary. After applying the gradient operator in spherical coordinates and taking on account the revolution symmetry, the torque component $\vec{M}_j$ reduces to

$$\vec{M}_j = r_G \int_{\mathcal{S}} \eta\, \vec{u} \wedge \left(\frac{\mu}{r^2} \sum_{l=2}^{\infty} J_l \left(\frac{R}{r}\right)^l \left[(l+1)\, P_l(\sin\varphi)\, \vec{u}_r - \frac{\mathrm{d}P_l(\sin\varphi)}{\mathrm{d}(\sin\varphi)} \cos\varphi\, \vec{u}_\varphi\right]\right) \mathrm{d}m.$$

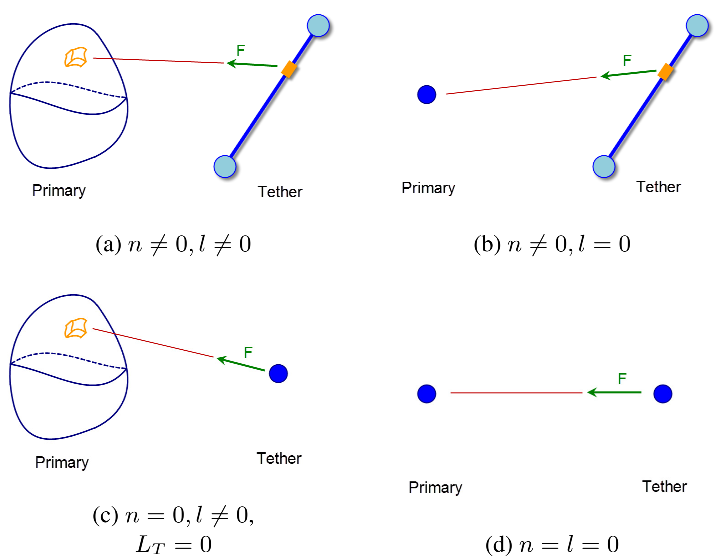
The derivation of the torque $\vec{M}_j$ into its final expression is exactly analogous to that of the gravitational resultant, $\vec{R}_j$, that we carried out in the previous section. The needed mathematical manipulations take nothing new that has not already been used or explained in the current paper, so we have omitted its full derivation for the sake of brevity, and decided to bring directly the final expression, assuming the reader should get to the same results with little effort. Thus, the torque component $\vec{M}_j$ finally takes the form

$$\begin{aligned} \vec{M}_j \quad = \quad & -\frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^{p} (-1)^n \, J_l \left(\frac{L_T}{r_G}\right)^{n+q+1} \left(\frac{R}{r_G}\right)^{l} b_{l-1,p}(3/2) \left(\begin{array}{c} p \\ q \end{array}\right) \cdot \\ & \cdot \sin^{p-q}(\varphi_G) \; (\vec{u}\cdot\vec{k}_4)^q \, C_n^{\left(\frac{l+p+2}{2}\right)}(\cos\alpha) \, a_{n+q+1} \left(\vec{u}\wedge\vec{k}\right) + \\ & + \frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l} \sum_{q=0}^{p} (-1)^n \, J_l \left(\frac{L_T}{r_G}\right)^{n+q+1} \left(\frac{R}{r_G}\right)^{l} b_{l,p}(1/2) \left(\begin{array}{c} p \\ q \end{array}\right) \cdot \\ & \cdot \sin^{p-q}(\varphi_G) \; (\vec{u}\cdot\vec{k})^q \, (l+1) \, C_n^{\left(\frac{l+p+3}{2}\right)}(\cos\alpha) \, a_{n+q+1} \; (\vec{u}\wedge\vec{u}_G) + \\ & + \frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^{p+1} (-1)^n \, J_l \left(\frac{L_T}{r_G}\right)^{n+q+1} \left(\frac{R}{r_G}\right)^{l} b_{l-1,p}(3/2) \left(\begin{array}{c} p+1 \\ q \end{array}\right) \cdot \\ & \cdot \sin^{p-q+1}(\varphi_G) \; (\vec{u}\cdot\vec{k})^q \, C_n^{\left(\frac{l+p+4}{2}\right)}(\cos\alpha) \, a_{n+q+1} \; (\vec{u}\wedge\vec{u}_G) \, , \end{aligned} \qquad (28)$$

and the total torque acting upon the tethered system, $\vec{M}$, is of course the vector sum of both contributions, $\vec{M}_s$ and $\vec{M}_j$.

FINAL EXPRESSIONS AND SIMPLIFICATIONS

Once the mutual gravitational potential V (Eqs. (12) + (24)), resultant $\vec{R}$ (Eqs. (20) + (27)) and torque $\vec{M}$ (Eqs. (21) + (28)) have been developed in the general approach of the Full Two-Body Problem (Figure 2a), one can easily simplify them in order to obtain any of the other approaches to the Two-Body Problem (Figures 2b - 2d) that we introduced in section .



(a) $n \neq 0, l \neq 0$ (b) $n \neq 0, l = 0$ (c) $n = 0, l \neq 0$, $L_T = 0$ (d) $n = l = 0$

Figure 5: **The general expressions for the full two-body problem can be simplified to provide expressions applicable to the other simpler approaches of the Two-Body Problem introduced in Fig. 2**

The general expressions are mainly double infinite series expansions with summation indexes n and l. The index l is related to the degree of accuracy with which the celestial body is modeled as an extensive body. Hence, truncating the summation at $l = 0$ would lead to the case of an extensive tether being attracted by a mass point celestial body, as in Figure 5b.

Similarly, the index n is related to the degree of accuracy with which the tether is modeled as an extensive body. Therefore, truncating the summation at $n = 0$ is expected to lead to the case of a mass point tether attracted by an extensive body, as sketched in Figure 5c, but it turns out that this simplification is not enough, but additionally one has to impose $L_T = 0$. We will try to further clarify this issue in the next section.

Evidently, truncating both series together at $n = 0$ and $l = 0$ would lead to the case in which both bodies are considered as mass points, as shown in Figure 5d.

A CLOSER LOOK AT THE GENERAL EQUATIONS

If we take a careful look at the final Equations we realize that summations over index n begin at $n = 0$ while summations over l begin at $l = 2$. This is because in their derivation we chose to break the equations down into 1) a *spherical* component, where the tether is assumed extensive but the primary is spherical, so there is only a summation over n, and 2) a component gathering the mixed effect of the extensive tether and zonal harmonics of the primary, hence with a double summation over n and l. Of course, as the first zonal harmonic is in fact the spherical case, both components can be brought together into more compact expressions, where the single summation over n is contained inside the double summation and indexes begin at $n = 0$ and $l = 0$. For the sake of example, let's take the gravitational potential V, which can be more compactly rewritten as

$$\begin{aligned} V \;=\; & +\frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{p=0}^{l} \sum_{q=0}^{p} (-1)^n \, a_{n+q} \, J_l \left(\frac{R}{r_G}\right)^l \left(\frac{L_T}{r_G}\right)^{n+q} b_{l,p}(1/2) \begin{pmatrix} p \\ q \end{pmatrix} \cdot \\ & \cdot \sin^{p-q}(\varphi_G) \cdot (\vec{u} \cdot \vec{k})^q \cdot C_n^{\left(\frac{l+p+1}{2}\right)}(\cos\alpha). \end{aligned} \tag{29}$$

If fact, it is easy to check that when $l = 0$, then

$$q = p = 0, \quad J_0 = -1, \quad b_{0,0}(1/2) = 1, \quad C_n^{\left(\frac{l+p+1}{2}\right)}(\cos\alpha) = P_n(\cos\alpha), \quad \text{and} \quad \begin{pmatrix} p \\ q \end{pmatrix} = 1,$$

so equation (29) reduces to Eq. (12), namely V_s.

Another interesting issue is the fact that these expressions are given in the form of two embedded convergent summations in powers of two small parameters, L_T/r_G and R/r_G, as well as combinations of both. This brings on the table the next question: how do those series converge, or how small are each of the terms of these expressions? Or most importantly, in order to achieve a required accuracy, which terms should we retain when truncating these series expansions? These questions have no easy general answer and should rather be looked into for each particular scenario, though for typical low Earth orbits and reasonable tether lengths one can expect that

$$\left(\frac{L_T}{r_G}\right) \ll \left(\frac{R}{r_G}\right),$$

so the series on n should converge faster. However, when we take into account that the parameter R/r_G is always accompanied by coefficients J_l, whose value is smaller for increasing l, then we have that, for a certain value of n, there is a value of l beyond which the previous relation inverts, i.e.

$$\forall n, \exists L \quad \text{such that} \quad \left(\frac{L_T}{r_G}\right)^n \gtrsim \left(\frac{R}{r_G}\right)^l J_l \quad \forall l > L$$

where the sign '$\gtrsim$' is used because the decrease in the values of the coefficients J_l is not necessarily monotonic.

On the one hand, these series convergence rates depend on the magnitude of the small parameters, which are functions of r_G, R and L_T, so they depend on the primary, the orbit and the length of the tether.

On the other hand, the relative magnitude of each of the terms in the summation also depends on further parameters, namely 1) the primary's gravitational model, through the J_l coefficients, 2) the orbit, through the latitude φ_G, 3) the attitude of the tether, through the angle α and unit vector $\vec{u}$, and 4) the mass geometry of the tether, through the functions $a_n(\phi, \Lambda)$.

Thus, it is clear that deciding where to truncate these series expansions is not easy to know in advance, since it is very problem-dependent. However, we could try to get some insight by neatly sorting these terms and taking a look at them. Equation (29) is more advantageous than its original form for this task. Let us rewrite Eq. (29) as

$$V = -\frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \Psi_{nl}$$

where Ψ_{nl}, easily obtainable by identifying terms in Eq. (29), are functions that can be orderly sorted in a squared array, as

		l				
		0	1	2	3	$\cdots$
n	0	1	0	$\Psi_{0,2}$	$\Psi_{0,3}$	$\cdots$
	1	0	0	$\Psi_{1,2}$	$\Psi_{1,3}$	$\cdots$
	2	$\Psi_{2,0}$	0	$\Psi_{2,2}$	$\Psi_{2,3}$	$\cdots$
	3	$\Psi_{3,0}$	0	$\Psi_{3,2}$	$\Psi_{3,3}$	$\cdots$
	$\vdots$	$\vdots$	0	$\vdots$	$\vdots$	$\ddots$

where one can observe that $\Psi_{1,0} = 0$ because $a_1 = 0$, and similarly the column $\Psi_{n,1}$ is always null, since $J_1 = 0$.

Attending at the expressions for Ψ_{nl}, it is straightforward to see that

$$\Psi_{n,0} = (-1)^n \left(\frac{L_T}{r_G}\right)^n a_n \, P_n(\cos\alpha),$$

as making $l = 0$ gets rid of the effect of the zonals, thus we recover the expressions for an extensive tether around a spherical primary. However, a curious happening that we noticed in the previous section is that, against intuition,

$$\Psi_{0,l} \neq -J_l \left(\frac{R}{r_G}\right)^l P_l(\sin\varphi_G),$$

so making $n = 0$ is not a sufficient condition to recover the case of a mass point tether around a non-uniform primary. Instead, one must also impose $L_T = 0$. Indeed,

$$\lim_{L_T \to 0} \Psi_{0,l} = -J_l \left(\frac{R}{r_G}\right)^l P_l(\sin\varphi_G), \qquad \forall l.$$

Also note that the last equation is satisfied when $\vec{u} \cdot \vec{k} = 0$ as well, which also makes the trick.

Roughly explained, one would expect that making $n = 0$ would reduce the effect of the zonal harmonics as affecting only the motion of the tether center of mass, G. However, there is some remaining contribution of the local gravity gradient that affects each end-mass and distributed mass element of the tether differently. Thus, the only way to make this contribution vanish, is by imposing the additional constraint $L_T = 0$, so that the tether really does simplify to a mass point.

Now we are in situation to shed some light on the issue of truncating the series, since now we can give values to these Ψ_{nl} functions, and see how big or small they are relatively. In order to do so, as Ψ_{nl} depend on so many other parameters, we have to begin by setting up some realistic scenario to fix the values of all these parameters.

For example, we may take a low Earth orbit of 400 km of altitude with a tether length $L_T = 10$ km. Thus, the small parameters will become

$$\left(\frac{L_T}{r_G}\right) = 0.000738, \qquad \left(\frac{R}{r_G}\right) = 0.94099$$

If the tether has symmetric end masses, then $a_n = 0$ for $n = 1, 3, 5, 7, ...$ so to make an illustrative example we shall take $m_1 = 400$ kg, $m_2 = 800$ kg and $m_T = 10$ kg. This gives $\phi = 54.65°$ and $\Lambda = 0.00826$. Lastly, in order to avoid that α, φ_G or $(\vec{u} \cdot \vec{k})$ become zero, we should choose an arbitrary orbit geometry, as for example $\alpha = 45°$, $\varphi_G = 45°$ and $(\vec{u} \cdot \vec{k}) = 1$, which sets the tether aligned with vector $\vec{k}$. For this scenario, the first values of Ψ_{nl} are shown in Table 1. The observation of these results in this particular scenario shows that the largest term that considers the extension of the tether is $\Psi_{2,0}$, whose order of magnitude is the same as for the orbital perturbation introduced by the zonals J_3, J_4, and already bigger than the J_5 perturbation.

We also see that the series in n is so much rapidly converging than in l, since this is a low Earth orbit with a short tether, so $(L_T/r) \ll (R/r) \sim 1$. However, it is interesting to see what happens when we move further away in the index l, i.e. when we take higher degree zonal harmonics. For instance, the main contributions to the perturbation of J_{11} is given by $\Psi_{0,11} = -2.709 \cdot 10^{-9}$, which is already of the same order of magnitude as the mixed terms $\Psi_{1,2}$ and $\Psi_{2,2}$. Also, the perturbation of J_{15} already gives $\Psi_{0,15} = -7.962 \cdot 10^{-11}$, whose order of magnitude coincides with those of $\Psi_{3,0}$ and $\Psi_{1,3}$. Hence, we observe that for high precision applications where many zonal harmonics are required in the computations, it becomes necessary, for error consistency, not also to include the effect of the tether extension through the terms $\Psi_{n,0}$, but also through some of the mixed terms $\Psi_{n,l}$, for $n \geqslant 1$, $l \geqslant 2$.

In order to see how these values of Ψ_{nl} change under different values of some of the parameters, we tried also a equatorial orbit with the tether still aligned with $\vec{k}$, i.e. we set the values $\alpha = 90°$, $\varphi_G = 0°$ and $\vec{u} \cdot \vec{k} = 1$. Results are gathered in Table 2. The most notorious phenomenon is that Ψ_{nl} with odd index n or l have suffered a considerable drop of several orders of magnitude, though still remain non-zero.

Another interesting experiment is to see what happens if we force the tether to be contained in the equatorial plane too ($\vec{u} \cdot \vec{k} = 0$), so that the whole motion is planar. Here we have too basic possibilities, one where the tether is perpendicular to the position vector $\vec{r}_G$ (i.e. $\alpha = 90°$), and another where the tether lies in the local vertical ($\alpha = 0°$). In both cases, making $\vec{u} \cdot \vec{k} = 0$ has vanished $\Psi_{n,l}$ for odd values of l. The difference can be seen in the rows, where for $\alpha = 90°$, the $\Psi_{n,l}$ with odd index n still contain the several orders of magnitude drop in their value. For $\alpha = 0°$, however, the convergence in n is smooth again, where $\Psi_{n,l}$ are increasingly smaller for higher values of n, but without showing any sudden drop. In fact, for this planar motion Eq. (29) can be reduced to

$$V = \frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (-1)^n \, a_n \, J_l \left(\frac{R}{r_G}\right)^l \left(\frac{L_T}{r_G}\right)^n P_0(0) \, C_n^{\left(\frac{l+1}{2}\right)}(\cos\alpha),$$

where we can clearly see that the value of the potential V is related to the angle α only through ultraspherical polynomials.

In order to explore what happens with the resultant of the torque, rather than the potential, one could proceed similarly, with the difference that these magnitudes are vectorial instead of scalar. For this equatorial planar case, for example, it is of particular interest to observe how the expression for the gravity torque simplifies to

$$\begin{aligned}\vec{M} = &- \frac{\mu m}{r_G} \sum_{n=0}^{\infty} (-1)^n \, a_{n+1} \left(\frac{L_T}{r_G}\right)^{n+1} \cdot \Bigg[C_n^{(3/2)}(\cos\alpha) + \\ &- \sum_{l=2}^{\infty} J_l \left(\frac{R}{r_G}\right)^l \left(b_{l,0}(1/2) \, (l+1) \, C_n^{\left(\frac{l+3}{2}\right)}(\cos\alpha) + b_{l-1,0}(3/2) \, C_n^{\left(\frac{l+4}{2}\right)}(\cos\alpha) \right) \Bigg] (\vec{u} \wedge \vec{u}_G) + \\ &+ \frac{\mu m}{r_G} \sum_{n=0}^{\infty} \sum_{l=2}^{\infty} (-1)^n \, a_{n+1} \left(\frac{L_T}{r_G}\right)^{n+1} \left(\frac{R}{r_G}\right)^l J_l \, b_{l-1,0}(3/2) \, C_n^{\left(\frac{l+2}{2}\right)}(\cos\alpha) \; (\vec{u} \wedge \vec{k}).\end{aligned}$$

		l					
		0	1	2	3	4	5
n	0	1	0	$-2.397 \cdot 10^{-04}$	$-3.730 \cdot 10^{-07}$	$-5.160 \cdot 10^{-07}$	$-6.316 \cdot 10^{-08}$
	1	0	0	$3.462 \cdot 10^{-09}$	$-1.346 \cdot 10^{-11}$	$-8.011 \cdot 10^{-12}$	$-1.841 \cdot 10^{-13}$
	2	$1.204 \cdot 10^{-07}$	0	$-1.642 \cdot 10^{-09}$	$4.249 \cdot 10^{-12}$	$9.142 \cdot 10^{-13}$	$-3.498 \cdot 10^{-13}$
	3	$-4.140 \cdot 10^{-11}$	0	$-9.383 \cdot 10^{-13}$	$4.821 \cdot 10^{-15}$	$3.556 \cdot 10^{-15}$	$1.376 \cdot 10^{-16}$

Table 1: **Values of Ψ_{nl} for the scenario under study, with $\alpha = 45°$, $\varphi_G = 45°$ and $\vec{u} \cdot \vec{k} = 1$**

		l					
		0	1	2	3	4	5
n	0	1	0	$4.793 \cdot 10^{-04}$	$-1.235 \cdot 10^{-15}$	$4.763 \cdot 10^{-07}$	$3.446 \cdot 10^{-16}$
	1	0	0	$-1.043 \cdot 10^{-27}$	$4.72 \cdot 10^{-27}$	$-4.835 \cdot 10^{-30}$	$-6.583 \cdot 10^{-28}$
	2	$-2.409 \cdot 10^{-07}$	0	$-3.463 \cdot 10^{-10}$	$-1.853 \cdot 10^{-15}$	$-5.736 \cdot 10^{-13}$	$2.584 \cdot 10^{-16}$
	3	$-2.175 \cdot 10^{-25}$	0	$-5.214 \cdot 10^{-28}$	$-1.191 \cdot 10^{-21}$	$-1.209 \cdot 10^{-30}$	$2.135 \cdot 10^{-33}$

Table 2: **Values of Ψ_{nl} for the scenario under study, with $\alpha = 90°$, $\varphi_G = 0°$ and $\vec{u} \cdot \vec{k} = 1$**

		l					
		0	1	2	3	4	5
n	0	1	0	$4.793 \cdot 10^{-04}$	0	$4.763 \cdot 10^{-07}$	0
	1	0	0	0	0	0	0
	2	$-2.409 \cdot 10^{-07}$	0	$-3.463 \cdot 10^{-10}$	0	$-5.736 \cdot 10^{-13}$	0
	3	$-2.175 \cdot 10^{-25}$	0	$-5.214 \cdot 10^{-28}$	0	$-1.209 \cdot 10^{-30}$	0

Table 3: **Values of Ψ_{nl} for the scenario under study, with $\alpha = 90°$, $\varphi_G = 0°$ and $\vec{u} \cdot \vec{k} = 0$**

		l					
		0	1	2	3	4	5
n	0	1	0	$4.793 \cdot 10^{-04}$	0	$-4.763 \cdot 10^{-07}$	0
	1	0	0	0	0	0	0
	2	$4.817 \cdot 10^{-07}$	0	$1.385 \cdot 10^{-09}$	0	$3.441 \cdot 10^{-12}$	0
	3	$2.342 \cdot 10^{-10}$	0	$1.123 \cdot 10^{-12}$	0	$3.904 \cdot 10^{-15}$	0

Table 4: **Values of Ψ_{nl} for the scenario under study, with $\alpha = 0°$, $\varphi_G = 0°$ and $\vec{u} \cdot \vec{k} = 0$**

We can see that for the case of a librating tether aligned with the local vertical, i.e. $\alpha = 0°$, we get $\vec{M} \cdot (\vec{u} \wedge \vec{u}_G) = 0$. On the other side, the torque component along vector $\vec{u} \wedge \vec{k}$ is only non-zero for odd values of l, where $b_{l-1,0}(1/2) \neq 0$. This means that odd zonal harmonics provoke a torque which is contained in the orbital plane, and hence the torque tends to take the tether end-masses out of the equatorial plane, hence making the local vertical configuration of the tether a non-equilibrium configuration. This is a remarkable fact, since in the in-plane orbital motion, the local vertical configuration is a equilibrium position of indifferent stability properties, and the system responds to a $\Delta\alpha$ perturbation by introducing a disturbing torque

$$\Delta\vec{M} = \frac{\mu m}{r_G} a_2 \left(\frac{L_T}{r_G}\right)^2 C_2^{(3/2)}(\cos\Delta\alpha) \; \sin\Delta\alpha \; \vec{k}$$

in its first order approximation, with $n = 2$ and $l = 0$. However, as soon as the tether leaves the plane, then $\vec{M} \cdot (\vec{u} \wedge \vec{u}_G) \neq 0$ and a restoring torque appears, yielding an oscillatory motions.

CONCLUSIONS

In this paper we have developed expressions for the gravitational actions (gravity potential V, resultant $\vec{R}$ and torque $\vec{M}$) upon a space tether under the restricted Full Two-Body approach, where the tether has been considered an extensive body with linear geometry, and the primary has been considered as an extensive body with revolution symmetry, where an arbitrary number of zonal harmonics have been retained. These expressions are given in the form of an embedded double series with two small parameters: a summation over index n in powers of L_T/r_G, which comes from the extension of the tether, and a summation over index l in powers of R/r_G, which comes from the zonal harmonics. We have studied through examples the convergence of these series for the gravitational potential, and concluded that for high precision applications where higher degree zonal harmonics need to be retained, then not only terms of the tether extension ($n \neq 0, l = 0$) must be taken into account, but also becomes necessary to consider some of the mixed terms ($n \neq 0, l \neq 0$) coming from the mutual two-body interaction, in order to keep consistency with the gravitational modeling of the dynamics.

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APPENDIX: ULTRASPHERICAL POLYNOMIALS

Ultraspherical polynomials are a generalization of Legendre polynomials.[12] They are orthogonal polynomials on the interval $[-1, 1]$, and are special cases of the Jacobi polynomials. They are solutions to the *Gengenbauer differential equation*

$$(1-\gamma^2)\,\frac{\mathrm{d}^2}{\mathrm{d}\gamma^2}C_n^{(\lambda)}(\gamma) - (2\lambda+1)\,\gamma\,\frac{\mathrm{d}}{\mathrm{d}\gamma}C_n^{(\lambda)}(\gamma) + n(n+2\lambda)\,C_n^{(\lambda)}(\gamma) = 0.$$

Therefore, these functions $C_n^{(\lambda)}(\gamma)$ are also known as *Gegenbauer Polynomials*. When $\lambda = 1/2$, the equation reduces to the Legendre equation, and the ultraspherical polynomials reduce to the Legendre polynomials.

These polynomials can also be defined in terms of their generating function, as the coefficients in a Taylor series expansion

$$\frac{1}{(1-2\,\gamma\,\eta+\eta^2)^\lambda} = \sum_{n=o}^{\infty} \eta^n\, C_n^{(\lambda)}(\gamma), \tag{30}$$

where $\eta \ll 1$.

Hence, the first ultraspherical polynomials are

$$\begin{aligned} C_0^{(\lambda)}(\gamma) &= 1 \\ C_1^{(\lambda)}(\gamma) &= 2\,\lambda\,\gamma \\ n\,C_n^{(\lambda)}(\gamma) &= 2\,(n+\lambda-1)\,\gamma\,C_{n-1}^{(\lambda)}(\gamma) - (n+2\,\lambda-2)\,C_{n-2}^{(\lambda)}(\gamma) \end{aligned}$$

Among many properties of the ultraspherical polynomials, the derivations in this paper make use of the following two

$$C_n^{(\lambda)}(-\gamma) = (-1)^n\, C_n^{(\lambda)}(\gamma) \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\, C_n^{(\lambda)}(\gamma) = 2\,\lambda\, C_{n-1}^{(\lambda+1)}(\gamma) \tag{32}$$

Ultraspherical polynomials, by definition, can be expressed as a summation of monomials. Thus, we could convert any $C_n^{(\lambda)}(\gamma)$ polynomial into a finite series by means of the relation

$$C_n^{(\lambda)}(\gamma) = \sum_{p=0}^{n} b_{np}(\lambda)\,\gamma^p \tag{33}$$

where the coefficients $b_{np}(\lambda)$ are easily identified.[13]