The extension of DROMO formulation to relative motion is evaluated. The orbit of the follower spacecraft can be constructed through differences on the elements defining the orbit of the leader spacecraft. Assuming that the differences are small, the problem is linearized. Typical linearized solutions to relative motion determine the relative state of the follower spacecraft at a certain time step. Because of the form of DROMO formulation, the performance of a frozen-anomaly transformation is explored. In this case, the relative state is computed for a certain value of the anomaly, equal for leader and follower. Since the time for leader and follower do not coincide, the implicit time delay needs to be corrected to recover the physical sense of the solution. When determining the relative orbit, numerical testing shows significant error reductions compared to previous linearized solutions.

INTRODUCTION

Relative motion is present in many situations in Space Dynamics. Modern applications include spacecraft formation flying, space debris active removal and collision avoidance, among others. These scenarios require an accurate model to describe the relative dynamics of the system. The present study focuses on the spacecraft formation flying problem.

Relative motion of the spacecraft is defined with respect to a certain reference within the formation, known as the leader (L). The leader is typically one spacecraft in the formation, but a virtual point is also a valid reference. The rest of the spacecraft in the formation are referred as followers (F). Relative motion is described by the motion of the followers around the leader spacecraft. Formulation here presented refers to the simplest case of formation flying, where only one follower spacecraft is considered.

The motion of two or more spacecraft following distinct orbits can be integrated directly. The relative state vector may be computed as the difference between the absolute states. This procedure leads to the exact solution to the problem. However, when applied to a generic case it implies certain numerical issues. First, the entire two-body problem needs to be integrated for each spacecraft. The computational cost of the integration grows with the number of spacecraft in the formation. Heavy computation is to be avoided when considering close-range maneuvers. Second, the solution lies on the least significant figures of the numerical subtraction. Relative distances are typically small when compared to the radius-vectors, leading to truncation errors and low accuracy. Hence, preferred methods are those that lead to the relative state directly.
First, a brief review of the existing linearized solutions to relative motion is presented. The governing equations of relative motion are formulated, introducing the notation to be used. Second, the new transformation concept is defined and DROMO formulation is described. Special attention is paid to the definition of the initial conditions and the existence of periodic solutions. Finally, the new method is tested through different numerical examples.

State of the art review

The first formal formulation of relative motion was provided by Hill. He posed the governing equations of relative motion when studying the dynamics of the Moon. Clohessy and Wiltshire linearized the equations of motion assuming small relative distances on a circular reference orbit, leading to the Hill-Clohessy-Wiltshire or Clohessy-Wiltshire equations. They provided an analytical solution for this particular case. However, this solution cannot be extended to generic elliptic orbits. Later efforts have been aimed to solve this particular problem.

Lawden found a closed form solution for the linearized relative motion on elliptic reference orbits while studying optimal trajectories. His solution is provided in terms of an integral, with no explicit solution. Tschauner and Hempel reduced the relative motion problem to a system of linear differential equations. This linear system governs the relative dynamics referred to elliptic reference orbits. Published in 1998, Carter found a singularity in Lawden’s solution. He then provided a state-transition matrix that depends explicitly on the true and the eccentric anomaly, overcoming the singularity. Yamanaka and Ankersen proposed an analytical solution to the Tschauner-Hempel equations. Their solution is given in the form of a time-explicit state-transition matrix.

Melton found a time-explicit state-transition matrix relying on an expansion in powers of the eccentricity. He adopted this approach to extend the classic formulation, based on circular reference orbits, to the elliptic case. In the work by Inalhan et al. a detailed study of periodicity conditions and initialization problems can be found.

Broucke built the relative orbit of the follower spacecraft through small differences on the classical elements defining the reference orbit. Resulting state-transition matrix is explicit in time. Variational formulation was also exploited by Lee et al. in 2007. They constructed a state-transition matrix based on a sensitivity matrix referred to the initial relative state vector. Relying on classical element differences, Schaub developed an anomaly explicit state-transition matrix. A particular geometric representation was adopted by Gim and Alfriend, finding a solution to the near-circular reference orbit subject to the \( J_2 \) perturbation.

The new method here presented is compared against the solutions proposed by Yamanaka and Ankersen, and by Schaub. Numerical integration of the Tschauner-Hempel equations is included in the comparison.

ON RELATIVE MOTION: DEFINING THE PROBLEM

The Geocentric Equatorial coordinate system is selected as the inertial reference \((1)\). Equations of motion are projected on the Euler-Hill rotating frame \((\mathcal{L})\), that follows the leader along its orbital motion. Additionally, the departure perifocal frame is introduced \((p)\). This frame corresponds to the perifocal frame at the initial time. Figure 1 sketches the different reference systems to be used.

Euler-Hill frame is centered on the leader spacecraft and defined by the basis \(\{i_{\mathcal{L}}, j_{\mathcal{L}}, k_{\mathcal{L}}\}\), being \(i_{\mathcal{L}}\) the versor along the radius-vector of the leader satellite. Versor \(k_{\mathcal{L}}\) is parallel to the angular momentum vector of the leader spacecraft and \(j_{\mathcal{L}}\) completes an orthonormal dextral frame.
Figure 1: Reference systems defined for the problem formulation

The two-body problem is governed by the equation:

\[ \frac{d^2 r}{dt^2} = -\frac{\mu}{|r|^3} r + a_p \]

Relative position vector is denoted by \( \rho = r_f - r_L \), where subindexes refer to the follower and leader spacecraft, respectively. The absolute acceleration for the follower spacecraft reads:

\[ \frac{d^2 r_f}{dt^2} = \frac{d^2}{dt^2} (r_L + \rho) = -\frac{\mu}{|r_L + \rho|^3} (r_L + \rho) + a_{p,f} \]

Subtracting the acceleration of the leader spacecraft, the relative acceleration results in:

\[ \frac{d^2 \rho}{dt^2} = \frac{\mu}{|r_L + \rho|^3} (r_L + \rho) + \frac{\mu}{|r_L|^3} r_L + a_{p,f} - a_{p,l}. \tag{1} \]

Time derivatives on the Euler-Hill rotating frame abide by:

\[ \frac{d^2 \rho}{dt^2} = \ddot{\rho} + \dot{\omega}_{L1} \times \rho + \omega_{L1} \times (\omega_{L1} \times \rho) + 2\omega_{L1} \times \dot{\rho}. \tag{2} \]

where \( \omega_{L1} \) is the angular velocity of the Euler-Hill frame w.r.t. the inertial reference. Projected on the Euler-Hill frame itself, \( \omega_{L1} \) is defined by:

\[ \omega_{L1} = \hat{\sigma} k_c = \frac{h}{|r_L|^2} k_c \]

Typical spacecraft formations satisfy that the relative distance is small compared to the orbital radius, i.e. \( |\rho| \ll |r_L| \). Under this assumption Equation (1) admits the following expansion:

\[ \frac{d^2 \rho}{dt^2} = \frac{\mu}{|r_L|^3} (3 \rho \circ [k_c, i_c] - \rho) + a_{p,rel}. \tag{3} \]
where \([i_L, i_L]\) refers to the diadic product. The term \(a_{p,rel} = a_{p,F} - a_{p,L}\) denotes the relative perturbation acceleration. Defining the relative position vector, \(\rho\), through its coordinates on the Euler-Hill frame, \(\rho = (x, y, z)^T\), Equations (2) and (3) lead to the system of linear differential equations proposed by Tschauner and Hempel:\(^4\)

\[
\begin{align*}
\ddot{x} - 2 \dot{\sigma} \dot{y} - \dot{\sigma}^2 x &= 2 \frac{\mu}{|r_L|^3} x + a_{p,rel} \cdot i_L \\
\ddot{y} + 2 \dot{\sigma} \dot{x} + \sigma x - \dot{\sigma}^2 y &= -\frac{\mu}{|r_L|^3} y + a_{p,rel} \cdot j_L \\
\ddot{z} &= -\frac{\mu}{|r_L|^3} z + a_{p,rel} \cdot k_L
\end{align*}
\]  

(4)  
(5)  
(6)

Assuming a circular reference orbit and neglecting external perturbations, these equations become the Clohessy-Wiltshire or CW equations:\(^2\)

\[
\begin{align*}
\ddot{x} - 2 n \dot{y} - 3 n^2 x &= 0 \\
\ddot{y} + 2 n \dot{x} &= 0 \\
\ddot{z} + n^2 z &= 0
\end{align*}
\]

A NOVEL TRANSFORMATION

The solutions discussed in the state of the art review compute the relative state vector at the same time step for leader and follower. Relative state vector is described by the change on the leader state vector due to the differences imposed to the reference orbit. These methods can be classified as frozen time transformations. There is no time delay associated to the transformation from the leader state vector to the relative state vector. In this work, the performance of a frozen anomaly transformation is analyzed. The precise definition of the anomaly is discussed in the following Section. The relative state is obtained for the same anomaly for leader and follower, but not at the same time. A certain time delay appears when applying a frozen anomaly transformation. Implicit time delay must be corrected a posteriori to recover the physical sense of the solution.

DROMO formulation

Relative motion is formulated through DROMO, propagator developed by Peláez et al.\(^13\) Integration is performed using the ideal anomaly, \(\sigma\), as the independent variable. The ideal anomaly is defined by the angle between the radius-vector and the initial eccentricity vector. If external perturbations are neglected, ideal and true anomaly coincide.

Variables in DROMO are dimensionless. A characteristic distance, \(L_c\), and a characteristic time, \(t_c\), are introduced. Dimensionless time is denoted by \(\tau = t/t_c\), and the rest of dimensionless variables are marked with \((\hat{\cdot})\).

Orbit geometry is described by the elements \(\zeta_1, \zeta_2\) and \(\zeta_3\). The set \((\zeta_1, \zeta_2)\) corresponds to the projections of the instantaneous eccentricity vector on the departure perifocal frame. For the unperturbed case, \(\zeta_1 = e\) and \(\zeta_2 = 0\). The term \(\zeta_3\) is the inverse of the dimensionless angular momentum, \(\zeta_3 = 1/\hat{h}\). The leader orbital plane is defined by the quaternion \(n = (\eta_1, \eta_2, \eta_3, \eta_4)\). The compo-
nents of the quaternion $n$ are related to the Euler angles through the expressions:

\[
\begin{align*}
\eta_1 &= \sin \frac{i}{2} \cos \frac{\Omega - \tilde{\omega}}{2}, \quad \eta_2 = \sin \frac{i}{2} \cos \frac{\Omega - \tilde{\omega}}{2} \\
\eta_3 &= \cos \frac{i}{2} \sin \frac{\Omega + \tilde{\omega}}{2}, \quad \eta_4 = \cos \frac{i}{2} \cos \frac{\Omega + \tilde{\omega}}{2}
\end{align*}
\]

verifying:

\[\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1\]

Observe that, in a generic case, the departure argument of the perigee, $\tilde{\omega}$, is different from the instantaneous argument of the perigee, $\omega$, since perturbation forces may modify the orbital plane.

Dimensionless position and velocity vectors are projected on the Euler-Hill frame as:

\[
\hat{r} = \frac{1}{\zeta_3 s} \hat{r}_L, \quad \hat{v} = \zeta_3 (u \hat{r}_L + s \hat{j}_L)
\]

being $s = 1 + \zeta_1 \cos \sigma + \zeta_2 \sin \sigma$ and $u = \zeta_1 \sin \sigma - \zeta_2 \cos \sigma$. Angular velocity reads:

\[
\frac{d\sigma}{d\tau} = \frac{\dot{h}}{\dot{r}^2} = \zeta_3 s^2
\]

Upon integration of the previous expression the time evolution is described by:

\[
\tau - \tau_0 = \int_{\sigma_0}^{\sigma} \frac{dq}{\zeta_3^2 (1 + \zeta_1 \cos \sigma + \zeta_2 \sin \sigma)^2}
\]

The time is a dependent variable in DROMO, defined by a function of the form $\tau = \tau(q; \sigma)$. The vector $q$ contains the DROMO elements and the initial condition, $\sigma_0$, being $q = (\zeta_1, \zeta_2, \zeta_3, n; \sigma_0)^T$.

**LINEARIZING THE EQUATIONS OF MOTION**

Equations of motion are linearized under the assumption that the difference between the DROMO elements defining the leader and the follower orbit is small. Follower orbit is constructed by applying a certain set of differences to the leader DROMO elements, $q_f = q_L + \delta q$. Relative position and velocity vectors correspond to the deviations on the reference state vector. Using dimensionless variables the transformation abides by:

\[
\hat{\rho} \equiv \delta \hat{r} = \frac{\partial \hat{r}_L}{\partial q} \delta q \quad \hat{\dot{\rho}} + \hat{\omega} L \times \hat{\rho} \equiv \delta \hat{v} = \frac{\partial \hat{v}_L}{\partial q} \delta q
\]

Hence, partial derivatives of Equation (7) w.r.t. the DROMO elements, $q$, lead to the relative state vector. Differential notation is convenient in this context. The dimensionless relative state vector, $\delta \hat{x}$, is defined as the deviation on the absolute state vector of the leader. For the sake of simplicity the components of the relative state vector will simply be denoted by $\delta \hat{x} = (\hat{x}, \hat{y}, \hat{z}, \hat{v}_x, \hat{v}_y, \hat{v}_z)^T$. 

5
Frozen-time transformation

From Equation (7) it is observed that \( r = r(q; \sigma) \) and \( v = v(q; \sigma) \). Additionally, Equation (8) describes the time as a function of the form \( \tau = \tau(q; \sigma) \). The inverse relation reads \( \sigma = \sigma(q; \tau) \).

When the time is frozen during the transformation, \( \tau_L = \tau_F = \tau^* \), the ideal anomaly verifies:

\[
\sigma_L = \sigma(q_L; \tau^*), \quad \sigma_F = \sigma(q_F; \tau^*)
\]

In a general case:

\[
\sigma_F - \sigma_L = \delta \sigma \neq 0
\]

Modifying the geometry of the orbit implies that, for the same time step, angular position along the orbit changes.

Frozen-anomaly transformation

Assume that the ideal anomaly remains constant through the transformation, \( \sigma_L = \sigma_F = \sigma^* \).

Equation (8) applied to both the leader and follower spacecraft yields:

\[
\tau_L = \tau(q_L; \sigma^*), \quad \tau_F = \tau(q_F; \sigma^*)
\]

A certain time delay appears:

\[
\tau_F - \tau_L = \delta \tau \neq 0
\]

This implicit time delay needs to be corrected a posteriori in order to recover the practical sense of the solution.

THE TRANSFORMATION

The dimensionless relative state vector has been defined as \( \delta \hat{x} = (\hat{x}, \hat{y}, \hat{z}, \hat{v}_x, \hat{v}_y, \hat{v}_z) \)\(^\top\). Linearized equations are written in matrix form:

\[
\delta \hat{x}(\sigma) = \mathcal{M}(q_L; \sigma) \delta q
\]

When no differences are applied to the leader orbit, \( \delta q = 0 \), the relative state vector is null and absolute states for leader and follower coincide. The transformation matrix \( \mathcal{M}(q_L; \sigma) \) computes the relative state vector given a certain set of differences on the DROMO elements, \( \delta q \). In order to define a bijective transformation of the form:

\[
\mathcal{M} : \mathbb{R}^8 \rightarrow \mathbb{R}^8
\]

two additional terms are added to the relative state vector. The first term is associated to the time delay:

\[
\delta \tau = \tau_F - \tau_L = \frac{\partial \tau}{\partial q} \delta q
\]

The derivatives of Equation (8) w.r.t. the DROMO elements result in:

\[
\delta \tau = \sum_{i=1}^{3} J_i \delta \zeta_i + J_4 \delta \sigma_0
\]
where the auxiliary terms $J_i$ are:

\[ J_1 = \int_{\sigma_0}^{\sigma} \frac{2 \cos \rho}{\zeta_3^2 s(q)^3} d\rho, \quad J_2 = \int_{\sigma_0}^{\sigma} \frac{2 \sin \rho}{\zeta_3^2 s(q)^3} d\rho, \]

\[ J_3 = \int_{\sigma_0}^{\sigma} \frac{3}{\zeta_3^2 s(q)^2} d\rho, \quad J_4 = \frac{1}{\zeta_3^4 s^2} - \frac{1}{\zeta_3^4 s_0^2} \]

An analytical solution for the integrals $J_1$, $J_2$ and $J_3$ is obtained for the unperturbed case.

The closing equation is derived from the condition for the quaternion $\mathbf{n}$ to be unitary:

\[ \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1 \Rightarrow \eta_1 \delta \eta_1 + \eta_2 \delta \eta_2 + \eta_3 \delta \eta_3 + \eta_4 \delta \eta_4 = 0 \quad (10) \]

Differences on the components of the quaternion $\mathbf{n}$, $\delta \eta_i$, are not independent. A degree of freedom is restricted. Adding both the term associated to the time delay and the condition on $\mathbf{n}$ to the relative state vector, it becomes $\delta \mathbf{x} = (\hat{x}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, 0, \delta \tau)^T$.

The transformation is detailed in Equation (11). The operator $\mathcal{M}$ is written in a compact way using the submatrices $\mathcal{A}$ and $\mathcal{B}$. These submatrices come from the partial derivatives of the position and velocity vectors w.r.t. the quaternion $\mathbf{n}$:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{\mathbf{v}}_x \\
\dot{\mathbf{v}}_y \\
\dot{\mathbf{v}}_z \\
\delta \tau
\end{bmatrix} =
\begin{bmatrix}
-\frac{\cos \sigma}{\zeta_3^2 s^2} & -\frac{\sin \sigma}{\zeta_3^2 s^2} & -\frac{2}{\zeta_3^3} & 0 & 0 & 0 & 0 & \frac{u}{\zeta_3^2 s^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2 \zeta_3 (u \mathcal{A} + s \mathcal{B})}{\zeta_3^3} & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\delta \zeta_1 \\
\delta \zeta_2 \\
\delta \zeta_3 \\
\delta \eta_1 \\
\delta \eta_2 \\
\delta \eta_3 \\
\delta \eta_4 \\
\delta \sigma_0
\end{bmatrix}
\]

being the auxiliary term $l = \sqrt{1 - \zeta_1^2 - \zeta_2^2}$ and the submatrices:

\[
\mathcal{A}^T = 
\begin{bmatrix}
0 & +\eta_2 & +\eta_3 \cos \sigma + \eta_4 \sin \sigma \\
0 & -\eta_1 & -\eta_4 \cos \sigma + \eta_3 \sin \sigma \\
0 & +\eta_4 & -\eta_1 \cos \sigma - \eta_2 \sin \sigma \\
0 & -\eta_3 & +\eta_2 \cos \sigma - \eta_1 \sin \sigma
\end{bmatrix},
\mathcal{B}^T = 
\begin{bmatrix}
-\eta_2 & 0 & +\eta_4 \cos \sigma - \eta_3 \sin \sigma \\
+\eta_1 & 0 & +\eta_3 \cos \sigma + \eta_4 \sin \sigma \\
-\eta_4 & 0 & -\eta_2 \cos \sigma + \eta_1 \sin \sigma \\
+\eta_3 & 0 & -\eta_1 \cos \sigma - \eta_2 \sin \sigma
\end{bmatrix}
\]

**Unperturbed case**

Neglecting external perturbations DROMO elements are constant during the propagation. Differences defining the follower orbit, $\delta q$, are equally constant. The initial state vector reads:

\[ \delta \mathbf{x}_0 = \mathcal{M}(\sigma_0) \delta q \]
Propagation can be defined in terms of the initial state vector:

$$\delta \hat{x}(\sigma) = \mathbf{M}(\sigma) \delta q = \mathbf{M}(\sigma) \mathbf{M}^{-1}(\sigma_0) \delta \hat{x}_0$$

The integrals \( J_1, J_2 \) and \( J_3 \) that appear from deriving Equation (9) admit an analytical solution of the form:

\[
J_1 = -\frac{3}{l^2} \zeta_1 (\tau - \tau_0) + \frac{1}{\zeta_3 l^2} \left( \frac{1 + s}{s^2} \sin \sigma - \frac{1 + s_0}{s_0^2} \sin \sigma_0 \right)
\]

\[
J_2 = 0
\]

\[
J_3 = \frac{3}{\zeta_3} (\tau - \tau_0)
\]

**Initial conditions**

Initial conditions can be defined in two different ways. i) Relative position and velocity vectors at departure time, \( \rho_0 \) and \( \dot{\rho}_0 \). ii) Differences on the DROMO elements describing the reference orbit, \( \delta q \).

The differential initial conditions are defined in a more intuitive way through the equivalent differences on the classical elements, \( \alpha = (a, e, i, \dot{\omega}, \Omega, M_0) \). In order to improve the versatility of the method, a linear transformation is defined to convert the differences on the classical elements to differences on the DROMO elements. The transformation reads:

$$\delta q = \mathbf{L}_{\mbox{CD}}(\alpha_L) \delta \alpha$$

where the matrix \( \mathbf{L}_{\mbox{CD}}(\alpha_L) \) is defined by the partial derivatives of the DROMO elements w.r.t. the classical elements:

$$\mathbf{L}_{\mbox{CD}} = \frac{\partial \mathbf{q}}{\partial \alpha}$$

Matrix formulation now admits the initial conditions to be defined using the classical elements:

$$\delta \hat{x} = \mathbf{M} \delta q = \mathbf{M} \mathbf{L}_{\mbox{CD}} \delta \alpha$$

The construction of matrix \( \mathbf{L}_{\mbox{CD}} \) is detailed in the Appendix.

**Correction of the time delay**

Equation (9) provides the time delay at every propagation step. A correction is applied to the solution, \( \hat{x}(\tau) \), so it is transformed into the solution at leader time, \( \hat{x}(\tau_L) \). The follower position vector corrected to the reference time, \( \hat{r}_F^* \), reads:

$$\hat{r}_F^* = \hat{r}_F(\tau_L) = \hat{r}_F(\tau_F - \delta \tau) + \hat{r}_L(\tau_L) + \dot{\rho}(\tau_F - \delta \tau)$$

Under the assumption that the time delay is small when compared to the characteristic time, the previous expression admits an asymptotic expansion of the form:

$$\hat{r}_F^* = \hat{r}_F(\tau_F) - \frac{d\hat{r}}{d\tau}\bigg|_{\tau=\tau_F} \delta \tau + \frac{d^2\hat{r}}{d\tau^2}\bigg|_{\tau=\tau_F} \delta \tau^2 + \mathcal{O}(\delta \tau^3)$$

$$\hat{v}_F^* = \hat{v}_F(\tau_F) - \frac{d\hat{v}}{d\tau}\bigg|_{\tau=\tau_F} \delta \tau + \mathcal{O}(\delta \tau^2)$$
where the position and velocity vectors at \( \tau = \tau_F \) correspond to the instantaneous values computed through the transformation defined in Equation (11). Neglecting higher order terms, the dimensional corrected solution becomes:

\[
\begin{align*}
\mathbf{r}_F^* &= \mathbf{r}_F(t_F) - \mathbf{v}_F(t_F) \Delta t - \frac{\mu \Delta t^2}{|\mathbf{r}_F(t_F)|^3} \mathbf{r}_F(t_F) \\
\mathbf{v}_F^* &= \mathbf{v}_F(t_F) + \frac{\mu \Delta t}{|\mathbf{r}_F(t_F)|^3} \mathbf{r}_F(t_F)
\end{align*}
\]

**Periodic solutions**

For the relative orbit to be closed, period commensurability is to be satisfied. Let \( T_1 \) and \( T_2 \) be the orbital periods for two given spacecraft. The periods are said to be \( m : n \) commensurable if:

\[
\frac{T_1}{T_2} = \frac{m}{n}, \quad m, n \in \mathbb{N}
\]

A special case in relative motion is when periods are \( 1 : 1 \) commensurable, i.e. the orbital periods are the same. To generate these orbits, differences on the reference elements must not alter the semi-major axis, \( \delta a = 0 \). This condition is studied through the equation of energy applied to the follower spacecraft:

\[
E = \frac{1}{2} |\mathbf{v}_F|^2 - \frac{\mu}{|\mathbf{r}_F|} = -\frac{\mu}{2a_F}
\]  

(12)

Note that energy is conserved when keplerian orbits are considered. Introducing the effect of the differences on the reference elements, Equation (12) becomes:

\[
E = \frac{1}{2} |\mathbf{v}_{L} + \delta \mathbf{v}|^2 - \frac{\mu}{|\mathbf{r}_{L} + \delta \mathbf{r}|} = -\frac{\mu}{2(a_{L} + \delta a)}
\]

Imposing that the semi-major axis does not change, the initial conditions must satisfy:

\[
\frac{|\mathbf{v}_{L} + \delta \mathbf{v}|^2}{2} - \frac{\mu}{|\mathbf{r}_{L} + \delta \mathbf{r}|} = -\frac{\mu}{2a_{L}}
\]  

(13)

The linear form of this equation, assuming that the differences are of the order of \( \varepsilon \ll 1 \), is:

\[
\frac{1}{2} |\mathbf{v}_{L}|^2 \left( 1 + 2 \frac{\delta \mathbf{v}}{|\mathbf{v}_{L}|} \right) - \frac{\mu}{|\mathbf{r}_{L}|} \left( 1 - \frac{\mathbf{r}_{L} \cdot \delta \mathbf{r}}{|\mathbf{r}_{L}|^2} \right) = -\frac{\mu}{2a_{L}} + \mathcal{O}(\varepsilon^2)
\]

Subtracting the equation of energy for the leader spacecraft leads to:

\[
\mathbf{v}_{L} \cdot \delta \mathbf{v} + \frac{\mu}{|\mathbf{r}_{L}|^3} (\mathbf{r}_{L} \cdot \delta \mathbf{r}) = \mathcal{O}(\varepsilon^2)
\]  

(14)

Neglecting second order terms, the linear condition on the initial relative state results in:

\[
\mathbf{v}_{L} \cdot \delta \mathbf{v} + \frac{\mu}{|\mathbf{r}_{L}|^3} (\mathbf{r}_{L} \cdot \delta \mathbf{r}) = 0
\]  

(15)

This result is the linearized condition on \( \delta \mathbf{r} \) and \( \delta \mathbf{v} \) for the relative orbit to be periodic.
By comparing Equations (14) and (15) it is observed that the initial conditions that lead to periodic solutions are different in the linear (Lin) and in the nonlinear (NL) formulation. The difference lies on higher order terms, being \( \delta r_{NL} - \delta r^{Lin} = \mathcal{O}(\varepsilon^2) \) and \( \delta v_{NL} - \delta v^{Lin} = \mathcal{O}(\varepsilon^2) \). If the initial conditions are such that the linear solution is periodic, the nonlinear solution will not be periodic, and viceversa.

The same discussion applies to the inverse problem. The dimensionless initial state vector of the follower spacecraft, \( \delta \hat{x}_0 \), can be obtained by solving the two-body problem nonlinear equations for \( q = q_{L} + \delta q_{NL} \). On the other hand, the initial conditions in the linear formulation read:

\[
\delta \hat{x}_0 = \mathcal{M} \delta q^{Lin}
\]

The two set of differences differ in the nonlinear terms, \( \delta q_{NL} - \delta q^{Lin} = \mathcal{O}(\varepsilon^2) \). In this work, the initial state vector, \( \delta \hat{x}_0 \), is computed from the exact nonlinear equations given by \( \delta q_{NL} \). In order to obtain the set of differences on the reference DROMO elements that leads to \( \delta \hat{x}_0 \) in the linear formulation, \( \delta q^{Lin} \), the following expression is applied:

\[
\delta q^{Lin} = \mathcal{M}^{-1} \delta \hat{x}_0
\]

**NUMERICAL TESTING**

In this Section the proposed transformation is compared against the solutions provided by Yamakawa and Ankersen, and Schaub.\(^6\),\(^11\) Numerical integration of the Tschauner-Hempel equations is included as an additional reference.\(^4\) Solutions are compared attending to the error in determining the relative orbit. The error is defined as the root mean square (RMS) error:

\[
\epsilon_{RMS} = \sqrt{(x_l - x_e)^2 + (y_l - y_e)^2 + (z_l - z_e)^2}
\]

between the linear orbit (\( l \)) and the exact solution to the problem (\( e \)). The exact solution corresponds to solving numerically the nonlinear two-body problem and subtracting the absolute states of leader and follower. Orbits are assumed to be keplerian.

| Table 1: Definition of the four cases for numerical testing |
|-----------------------------|-------------|-------------|-------------|-------------|
| **UNITS** | **CASE 1** | **CASE 2** | **CASE 3** | **CASE 4** |
| \( a \) | km | 7642 | 11254 | 7555 | 74390 |
| \( e \) | ° | 0.10 | 0.30 | 0.13 | 0.80 |
| \( i \) | ° | 30 | 60 | 48 | 60 |
| \( \dot{\omega} \) | ° | 0 | 20 | 10 | 20 |
| \( \Omega \) | ° | 0 | 0 | 20 | 30 |
| \( M_0 \) | ° | 37.3 | 0.0 | 0.0 | -2.5 |
| \( x_0 \) | m | -10 | -100 | -7228 | -80 |
| \( y_0 \) | m | 100 | 10 | 19128 | 0.8 |
| \( z_0 \) | m | -10 | -500 | -8262 | -300 |
| \( \dot{x}_0 \) | m/s | -0.1 | -0.1 | -24.1 | -0.06 |
| \( \dot{y}_0 \) | m/s | 0.1 | 0.3 | 8.0 | 0.002 |
| \( \dot{z}_0 \) | m/s | -0.1 | -0.1 | -2.8 | -0.01 |
Four different cases are studied, summarized in Table 1. For each case the reference orbit is defined using the classical elements and the initial relative state vector is provided in terms of $\rho_0 = (x_0, y_0, z_0)^T$ and $\dot{\rho}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0)^T$. These vectors are projected on the Euler-Hill rotating frame.

The selected characteristic distance, $L_c$, and characteristic time, $t_c$, are $L_c = a$ and $t_c = 1/n$. Case 1 is similar to that solved by Yamanaka and Ankersen, and Case 3 reproduces the problem solved by Schaub. Cases 2 and 4 are examples of a mid and a high eccentricity reference orbit, respectively. Simulations are performed for four complete revolutions.

Figure 2 shows the relative orbits for Cases 1 and 2, projected on the Euler-Hill frame. Leader spacecraft is located at the origin. The exact relative orbit and the orbit obtained through the new transformation are displayed together with the solutions provided by Yamanaka and Ankersen, and

<table>
<thead>
<tr>
<th>Schaub</th>
<th>Y&amp;A</th>
<th>T&amp;H</th>
<th>New</th>
<th>Exact</th>
</tr>
</thead>
</table>

Figure 2: Relative orbits for Cases 1 and 2 projected on the Euler-Hill frame. The solutions proposed by Schaub (‘Schaub’), Yamanaka and Ankersen (‘Y&A’) and the integration of the Tschauner-Hempel equations (‘T&H’) are compared against the proposed solution (‘New’) and the exact solution (‘Exact’).
Schaub’s methods. Tschauner-Hempel equations are integrated numerically using a RK4 schema. Relative orbits are projected on the $x_L y_L$ plane and on the $y_L z_L$ plane. The projections correspond to the in-plane and out-of-plane motion, respectively, referred to the leader orbital plane. No difference between the solutions can be observed in this Figure.

Relative orbits corresponding to Cases 3 and 4 are presented in Figure 3. Initial conditions for these Cases lead to large relative distances when compared to Cases 1 and 2. In Figure 3a it is observed that the new solution is more accurate when defining the relative orbit. DROMO formulation and the frozen-anomaly transformation are more robust when facing nonlinearities. The other three linear solutions fail to determine the relative orbit. In this Case $|\rho|/|r_L| \sim 8\%$, the basic assumption $|\rho| \ll |r_L|$ is not satisfied and the linear methods diverge from the exact solution.

![Figure 3](image-url)

Figure 3: Relative orbits (Cases 3 and 4) projected on the $x_L y_L$ plane and on the $y_L z_L$ plane
The error for each Case is displayed in Figure 4. Figures 4a and 4b present the RMS errors for Cases 1 and 2. The Schaub, and Yamanaka and Ankersen’s solutions, and the numerical integration of Tschauner-Hempel equations provide similar results, since they rely on the same frozen-time concept. These methods lead to a maximum error for Case 1 of 2.5 m after four revolutions, being the relative error 0.04% when compared to the maximum relative distance. The error for the new

![Figure 4](image_url)

**Figure 4:** RMS error associated to the determination of the relative orbit through different methods. The error is plotted in polar coordinates, being the ideal anomaly of the leader spacecraft, $\sigma$, the polar angle.
solution is approximately 0.5 m, resulting in an error reduction of up to 80% compared to the other methods. For Case 2 the error reduction is around 90%, being reduced from a maximum of 50 m to 6 m.

Figures 4c and 4d plot the RMS errors for Cases 3 and 4. The errors are of several kilometers, because the differences on the reference elements are not small. In Case 3 the maximum error for the frozen-time solutions is 64 km, around 6% of the relative distance. This error is large since the condition for linearizing the equations of motion, $|\rho| \ll |r_L|$, is not satisfied. The maximum error associated to the proposed solution is 25 km, exhibiting a 60% error reduction. In Case 4, an example of a high eccentricity reference orbit, the error for the new solution remains below 150 m (0.03%), whereas the maximum error for the rest of methods is around 6.4 km. An error reduction of up to 95% is observed in this Case. Figure 4d includes a zoomed view of the error.

**CONCLUSIONS**

DROMO formulation has been successfully extended to the relative motion problem. This propagator exhibits numerical advantages and improved performance when compared to equivalent methods. Its properties are not depreciated when linearized. A new transformation has been presented, relying on the concept of a frozen-anomaly transformation.

The proposed solution has been compared against several reference methods by solving four numerical Cases. Important error reductions are observed (between 60% and 95%) when computing the relative orbit. The main drawback of this formulation is the need of an additional transformation to correct the time delay. However this extra source of error, the overall performance is not affected. The proposed solution is more robust when facing nonlinear terms.

For simplicity the formulation has been applied to a two-spacecraft formation. The extension to multiple-spacecraft formations is immediate, as well as the extension to any other relative motion scenarios. The solution has been restricted to the unperturbed motion. Future actions include adding an adequate perturbation model to the formulation.

**ACKNOWLEDGEMENT**

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**REFERENCES**


APPENDIX: CLASSICAL ELEMENTS TO DROMO ELEMENTS

Special attention has been paid to how the initial conditions are defined. In order to convert the differences on the classical elements to differences on the DROMO elements, the matrix $L_{CD}$ is introduced. This matrix defines a bijective transformation of the form:

$$
\delta q = L_{CD}(\alpha_L) \delta \alpha
$$

Note that the dimensions of the vector $\delta \alpha = (\delta a, \delta e, \delta i, \delta \dot{\omega}, \delta \Omega, \delta M_0)^T$ and the equivalent DROMO elements $\delta q = (\delta \zeta_1, \delta \zeta_2, \delta \zeta_3, \delta n; \delta \sigma_0)^T$ do not coincide. To obtain a regular transformation, two additional terms are added to $\delta \alpha$. The first term corresponds to the condition on the quaternion $n$ defined by Equation (10). The second term is given by the transformation $(\zeta_1, \zeta_2) \leftrightarrow (e, \beta)$, where $\beta$ defines the evolution of the eccentricity vector with respect to the initial eccentricity vector. The vector $\delta \alpha$ is extended to $\delta \alpha = (\delta a, \delta e, \delta i, \delta \omega, \delta \Omega, \delta M_0, 0, \delta \beta)^T$. Introducing the polar angle $\beta$ the difference between the departure and the instantaneous argument of the perigee reads:

$$
\omega = \tilde{\omega} + \beta
$$

Assuming the orbits to be keplerian $\beta = 0$, the eccentricity vector is invariant, $\zeta_1 = e$ and $\omega \equiv \tilde{\omega}$. Defining the auxiliary terms:

$$
\begin{align*}
  s_i &= \frac{1}{2} \sin \frac{i}{2}, & so^+ &= \sin \frac{\Omega + \tilde{\omega}}{2}, & co^+ &= \cos \frac{\Omega + \tilde{\omega}}{2} \\
  c_i &= \frac{1}{2} \cos \frac{i}{2}, & so^- &= \sin \frac{\Omega - \tilde{\omega}}{2}, & co^- &= \cos \frac{\Omega - \tilde{\omega}}{2}
\end{align*}
$$
the transformation matrix $\mathbf{L}_{\text{CD}}$ is defined as:

$$
\mathbf{L}_{\text{CD}} =
\begin{bmatrix}
0 & \cos \beta & 0 & 0 & 0 & 0 & 0 & -e \sin \beta \\
0 & e \sin \beta \cos \beta & 0 & 0 & 0 & 0 & 0 & e^2 \cos^2 \beta \\
-l_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a e}{h^3} & 0 & +ci co^- & -si so^- & +si so^- & 0 & +si co^- & -si so^- \\
0 & 0 & +ci so^- & +si co^- & -si co^- & 0 & +si so^- & +si co^- \\
0 & 0 & -si so^+ & +ci co^+ & +ci co^+ & 0 & +ci so^+ & +ci co^+ \\
0 & 0 & -si co^+ & -ci so^+ & -ci so^+ & 0 & +ci co^+ & -ci so^+ \\
0 & (2 + e \cos \sigma_0) \frac{\sin \sigma_0}{l^2} & 0 & 0 & 0 & \frac{a^2 l}{r_0^2} & 0 & 0
\end{bmatrix}
$$