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EDROMO: AN ACCURATE PROPAGATOR FOR ELLIPTICAL ORBITS IN THE PERTURBED TWO-BODY PROBLEM

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EDromo is a special perturbation method for the propagation of elliptical orbits in the perturbed two-body problem. The state vector consists of a time-element and seven spatial elements, and the independent variable is a generalized eccentric anomaly introduced through a Sundman time transformation. The key role in the derivation of the method is played by an *intermediate* reference frame which enjoys the property of remaining fixed in space as long as perturbations are absent. Three elements of EDromo characterize the dynamics in the *orbital* frame and its orientation with respect to the intermediate frame, and the Euler parameters associated to the intermediate frame represent the other four spatial elements. The performance of EDromo has been analyzed by considering some typical problems in astrodynamics. In almost all our tests the method is the best among other popular formulations based on elements.

INTRODUCTION

In 2000 a house-made orbital propagator for the perturbed two-body problem was developed by the Space Dynamics Group of the Universidad Politécnica de Madrid (former Grupo de Dinamica de Tethers) based on a set of redundant variables including Euler angles. The propagator was called Dromo[*] and it was mainly used in numerical simulations of electrodynamic tethers. The special perturbation method Dromo was presented for the first time in 2002 ([1]). A more detailed description of Dromo took place in the 15th AAS/AIAA Space Flight Mechanics Meeting in 2005 ([2]) and in the paper ([3]).

The Dromo propagator - in the version described in ref. ([3]) - consists of eight ordinary differential equations (ODEs) and the independent variable is a fictitious time which reduces to the true anomaly when the motion is unperturbed. Seven dependent variables are elements, that is, they are prime integrals of the Keplerian motion. The method as presented in ref. ([3]) is explained starting from a decomposition of the dynamics into the radial motion and the rotation of the radial direction in space. This approach which is sometimes called *projective decomposition* ([4]) leads to a set of four linear second-order differential equations for the inverse of the orbital radius and the unit position vector when the independent variable is the true anomaly. The Dromo method relies on elements

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[*]The word *dromo* is derived from the old Greek word δρόμος (dròmos) that means *running*.

that are strictly related to these variables, which are also known as Burdet-Ferrándiz focal variables after the methods described in the papers ([5]) and ([6]). More specifically, two elements are linked to the radial motion, one is the inverse of the angular momentum and the remaining four elements are the components of a unit quaternion which defines a reference frame linked to the *orbital* frame.

The main advantages of the Dromo method ([3]) are: unique formulation for elliptic, parabolic and hyperbolic orbits; the truncation error nearly disappears in the unperturbed problem and is scaled by the perturbation itself in the perturbed one; the Euler parameters avoid singularities and give easy auto-correction as well as robustness; easy programming; finally, it is not necessary to solve Kepler's equation in the elliptic case, nor the equivalent for hyperbolic and parabolic cases, since time is one of the dependent variables. The method has been recently improved in both accuracy and computational cost by the employment of a *generalized* Sundman time transformation which enables the introduction into the ODEs of the disturbing potential energy ([7]), and by the replacement of the physical time with a time-element ([8]).

Deprit in the paper ([9]) develops a set of elements almost identical to the Dromo elements by following a different derivation from Peláez et al. ([3]). The cornerstone in his approach is represented by the *ideal* reference frame which lies on the orbital plane and has the property of keeping unchanged its orientation as long as the perturbations are either absent or locked within the orbital plane. Such frame individuates a direction termed *departure* point from which the angles usually referred to the osculating eccentricity vector can be always reckoned even in the case of circular motion. After introducing Euler parameters to trace the evolution of the ideal frame, the dynamics with respect to this frame is completely characterized by the orbital angular momentum and the two projections of the Laplace vector along the axes of the ideal frame. Because the adopted independent variable is the physical time an additional element is required to fix the position along the osculating orbit. To this purpose the mean anomaly reckoned from the departure point is introduced and the corresponding differential equation is derived from the Kepler's equation written for elliptical motion. In the Dromo method this issue does not arise because the relative rotation between the ideal and the orbital frames is given by the independent variable itself.

Following the works of Deprit ([9]), Pelaez et al. ([3]) and Baù et al. ([7]) we propose eight elements devoted to the propagation of elliptical orbits. The resulting method, named EDromo, exploits a time transformation of the Sundman type involving the total energy to define a generalized eccentric anomaly as independent variable. First the equation of motion along the radial direction is regularized by *embedding* the total energy and then the method of variation of parameters is applied to produce two orbital elements. These new quantities are recognized as the projections of a generalized eccentricity vector along two orthogonal axes on the orbital plane, thus suggesting the introduction of an *intermediate* reference frame, analogous to the ideal frame in papers ([9]) and ([3]). The dynamics in the orbital frame and its orientation with respect to the intermediate frame can be described by the two elements together with the total energy, which are selected as dependent variables of the method. Four further elements are the components of a unit quaternion which allow to assess the attitude of the intermediate frame with respect to a *fixed* frame. Finally, in accordance with the definition given by Stiefel and Scheifele ([10] p. 83), a time-element is employed in order to compute the physical time when necessary.

The EDromo method is compared to other formulations which rely on elements for the propagation of perturbed elliptical motion around the Earth. As performance indicator we consider the accuracy in the position achieved at a desired epoch versus the number of evaluations of the vector field. In particular, we examine the problem of an initially highly eccentric orbit (eccentricity 0.95) perturbed by both the J_2 zonal harmonic and the Moon's gravitational attraction. This example, which has been extensively exploited in the literature, was created by Stiefel and Scheifele ([10] Section 23) to exalt their propagation scheme. In the results the reader can appreciate the outstanding performance of EDromo.

DYNAMICS IN THE ORBITAL REFERENCE FRAME

Let the state of a point mass be described by its position $\mathbf{r}$ and velocity $\mathbf{v}$ with respect to the central body and in a reference frame with fixed axes in space. We will adopt throughout this paper non-dimensional quantities such that the gravitational parameter of the primary body is equal to 1. The Newtonian equation yields:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} = \mathbf{v} \,, \tag{1}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} = -\frac{1}{r^2}\,\mathbf{r} + \mathbf{F} \,, \tag{2}$$

where $\mathbf{F}$ is the perturbing force.

The orbital angular momentum is then given by:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \,. \tag{3}$$

Let r and h be the magnitudes of the vectors $\mathbf{r}$ and $\mathbf{h}$ respectively, we define the *orbital* reference frame by means of the orthonormal basis ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$):

$$\mathbf{k} = \frac{\mathbf{h}}{h} \,, \qquad \mathbf{i} = \frac{\mathbf{r}}{r} \,, \qquad \mathbf{j} = \mathbf{k} \times \mathbf{i} \,. \tag{4}$$

The perturbation $\mathbf{F}$ is then regarded as the sum of its components along the axes of the orbital frame:

$$\mathbf{F} = R\,\mathbf{i} + T\,\mathbf{j} + N\,\mathbf{k} \,. \tag{5}$$

From Eq. (1), Eq. (3) and the first two relations in (4) the velocity is written as:

$$\mathbf{v} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\,\mathbf{i} + \frac{h}{r}\,\mathbf{j} \,. \tag{6}$$

Differentiation of Eq. (3) with respect to time yields:

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\tau} = \mathbf{r} \times \mathbf{F} \,, \tag{7}$$

from which the time-derivative of h becomes:

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = (\mathbf{r} \times \mathbf{F}) \cdot \mathbf{k} = r\,T \,. \tag{8}$$

The evolution of the orbital frame is governed by the first-order differential equations:

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}\tau} = \mathbf{w}_o \times \mathbf{k} \,, \qquad \frac{\mathrm{d}\mathbf{i}}{\mathrm{d}\tau} = \mathbf{w}_o \times \mathbf{i} \,, \qquad \frac{\mathrm{d}\mathbf{j}}{\mathrm{d}\tau} = \mathbf{w}_o \times \mathbf{j} \,, \tag{9}$$

where the angular velocity $\mathbf{w}_o$ is obtained from the first two relations in (9) by exploiting Eqs. (4) and (6) - (8):

$$\mathbf{w}_o = N\,\frac{r}{h}\,\mathbf{i} + \frac{h}{r^2}\,\mathbf{k}\,. \tag{10}$$

After this introduction we deal more specifically with the dynamics on the orbital plane. We plug the expression for $\mathbf{v}$ given in (6) into Eq. (2), carry out the derivative and project along the radial direction to get:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{h^2}{r^3} - \frac{1}{r^2} + R\,. \tag{11}$$

The first step towards the regularization of Eq. (11) is to introduce a new independent variable by means of the Sundman's time transformation in the form:

$$\frac{\mathrm{d}\tau}{\mathrm{d}u} = r\,\sqrt{a}\,, \tag{12}$$

where the quantity a is the semi-major axis of the osculating orbit and is defined by:

$$\frac{1}{a} = -\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \left(\frac{h}{r}\right)^2 + \frac{2}{r}\,. \tag{13}$$

The fictitious time u coincides with the eccentric anomaly unless an arbitrary constant. In order to write the left-hand side of Eq. (11) with respect to the new independent variable we need the relation:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{1}{r\sqrt{a}}\,\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{r\sqrt{a}}\,\frac{\mathrm{d}r}{\mathrm{d}u}\right),$$

which brings to the formula:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{1}{a\,r^2}\left(\frac{\mathrm{d}^2 r}{\mathrm{d}u^2} - \frac{1}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)^2 - \frac{1}{2\,a}\,\frac{\mathrm{d}r}{\mathrm{d}u}\,\frac{\mathrm{d}a}{\mathrm{d}u}\right). \tag{14}$$

Before employing Eq. (14) into Eq. (11) we operate the substitution:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)^2 = 2\,a\,r - r^2 - a\,h^2\,,$$

which follows from Eq. (13) with the aid of Eq. (12). Finally, after some simplifications Eq. (11) is transformed into:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}u^2} + r - a = a\,r^2 R + \frac{1}{2\,a}\,\frac{\mathrm{d}r}{\mathrm{d}u}\,\frac{\mathrm{d}a}{\mathrm{d}u}\,, \tag{15}$$

where the terms related to the perturbations have been moved to the right-hand side. Indeed, differentiation of Eq. (13) with respect to τ, subsequent insertion of Eqs. (11), (8) and (6), and final switch to u through Eq. (12) lead to the result:

$$\frac{\mathrm{d}a}{\mathrm{d}u} = 2\,a^{5/2} r\,(\mathbf{F}\cdot\mathbf{v})\,. \tag{16}$$

We note that by introducing a fictitious time u through Eq. (12) and exploiting the integral of the Keplerian energy (13) we achieve the regularization of Eq. (11) at least in the unperturbed part. This well-known result represents the starting point of our presentation of a new set of differential equations to describe the perturbed two-body problem.

Orbital elements

Let us assume that the perturbing force $\mathbf{F}$ is zero, that is $R = T = N = 0$. Then, the differential equation (15) takes the form of an harmonic oscillator:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}u^2} = -r + a \tag{17}$$

perturbed by the quantity a, which according to Eq. (16) and our assumption is a constant. Equation (17) is analytically integrable and the solution results:

$$r = a\,(1 - A_1 \cos u - A_2 \sin u)\,,$$

where A_1 and A_2 are constants of integration.

In order to tackle the perturbed problem we write the orbital radius in the form:

$$r = \eta_3\,(1 - \eta_1 \cos u - \eta_2 \sin u)\,, \tag{18}$$

where $\eta_1(u)$, $\eta_2(u)$, and $\eta_3(u)$ are regarded as spatial dependent variables. In particular η_3 coincides with the osculating semi-major axis. For convenience we introduce the auxiliary quantity:

$$\rho = 1 - \eta_1 \cos u - \eta_2 \sin u\,. \tag{19}$$

The method of *variation of parameters* is used to derive the differential equations of η_1 and η_2. The osculating condition:

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}u}\cos u + \frac{\mathrm{d}\eta_2}{\mathrm{d}u}\sin u = \frac{\mathrm{d}\eta_3}{\mathrm{d}u}\,\frac{\rho}{\eta_3} \tag{20}$$

assures that the radial velocity takes always the Keplerian form:

$$\frac{\mathrm{d}r}{\mathrm{d}u} = \eta_3\,(\eta_1 \sin u - \eta_2 \cos u)\,. \tag{21}$$

The latter equation is differentiated with respect to u and the resulting expression along with Eq. (21) are plugged into Eq. (15) which converts into:

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}u}\sin u - \frac{\mathrm{d}\eta_2}{\mathrm{d}u}\cos u = \eta_3^2\,\rho^2 R - \frac{1}{2\,\eta_3}\,\frac{\mathrm{d}\eta_3}{\mathrm{d}u}\,(\eta_1 \sin u - \eta_2 \cos u)\,. \tag{22}$$

Equations (20) and (22) are solved for the two unknowns. The u-derivative of η_3 is provided by Eq. (16) where a is replaced with η_3, and $\mathbf{F}$, $\mathbf{v}$ and r by means of Eqs. (5), (6) and (18) respectively. The angular momentum enters in the transverse component of $\mathbf{v}$ and is written as:

$$h = \sqrt{\eta_3}\,l\,, \tag{23}$$

where:

$$l = \sqrt{1 - e^2}\,, \tag{24}$$

and e is the osculating eccentricity, which as will be shown in the next section depends only on η_1 and η_2. Note that for a finite value of a the quantity l can take any value in the range $[0,\,1)$. The differential equations of the three orbital elements are:

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}u} = \eta_3^2\left\{R\left(l^2 \sin u - 2\,\rho\,\eta_2\right) + T\,l\,[(1+\rho)\cos u - \eta_1]\right\}, \tag{25}$$

$$\frac{\mathrm{d}\eta_2}{\mathrm{d}u} = \eta_3^2\left\{R\left(2\,\rho\,\eta_1 - l^2 \cos u\right) + T\,l\,[(1+\rho)\sin u - \eta_2]\right\}, \tag{26}$$

$$\frac{\mathrm{d}\eta_3}{\mathrm{d}u} = 2\,\eta_3^3\,[R\,(\eta_1 \cos u - \eta_2 \sin u) + T\,l]\,, \tag{27}$$

and the time-transformation (12) becomes:

$$\frac{\mathrm{d}\tau}{\mathrm{d}u} = \eta_3^{3/2} \rho \, . \tag{28}$$

THE INTERMEDIATE REFERENCE FRAME

The independent variable u is strictly related to the eccentric anomaly E. Indeed, by comparison of Eq. (18) with:

$$r = a\,(1 - e\cos E)\,, \tag{29}$$

one infers the following relations for η_1 and η_2:

$$\eta_1 = e\cos\alpha\,, \qquad \eta_2 = e\sin\alpha\,, \qquad \alpha = u - E\,. \tag{30}$$

The value of u at $\tau = 0$ can be chosen as:

$$u_0 = E_0\,,$$

so that we have $\alpha_0 = 0$ and $\eta_{1,0} = e_0$, $\eta_{2,0} = 0$. Because the eccentricity vector $\mathbf{e}$ lies on the orbital plane by definition:

$$\mathbf{e} = -\mathbf{i} - \mathbf{h} \times \mathbf{v}\,, \tag{31}$$

there exist two orthonormal vectors $\mathbf{x}$ and $\mathbf{y}$ on the orbital plane such that η_1 and η_2 are the components of $\mathbf{e}$ along these vectors:

$$\mathbf{e} = \eta_1\,\mathbf{x} + \eta_2\,\mathbf{y}\,. \tag{32}$$

Let us introduce the orthonormal basis $(\mathbf{x},\,\mathbf{y},\,\mathbf{k})$, and call it the *intermediate* reference frame. The radial and transverse unit vectors of the orbital frame can be obtained by the rotation (see Fig. 1):

$$\mathbf{i} = \mathbf{x}\cos\nu + \mathbf{y}\sin\nu\,, \tag{33}$$

$$\mathbf{j} = -\mathbf{x}\sin\nu + \mathbf{y}\cos\nu\,, \tag{34}$$

$$\nu = f + \alpha\,, \tag{35}$$

where f is the osculating true-anomaly. As a consequence the angular velocities of the intermediate frame with respect to the orbital and fixed frames are respectively:

$$\mathbf{w}_{io} = -\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\,\mathbf{k}\,, \tag{36}$$

$$\mathbf{w}_i = \mathbf{w}_o + \mathbf{w}_{io} = N\,\frac{r}{h}\,\mathbf{i} + \omega\,\mathbf{k}\,, \qquad \omega = \frac{h}{r^2} - \frac{\mathrm{d}\nu}{\mathrm{d}\tau}\,, \tag{37}$$

where $\mathbf{w}_o$ is reported in Eq. (10). When the motion is purely Keplerian the intermediate frame remains fixed in space, but in general its attitude is influenced by any of the components R, T, and N of the perturbation. On the other hand the *ideal* frame employed by Deprit ([9]) and Peláez et al. ([3]) does not change its orientation even in presence of R and T.

At this point it is clear that if we know the variables η_1, η_2 and η_3 and the evolution of the intermediate frame we can fully determine the state of the propagated point mass.

An explicit expression for ω has not been provided yet. To this end let us first define the *apsidal* reference frame by the unit vectors:

$$\mathbf{a} = \frac{\mathbf{e}}{e}\,, \qquad\qquad \mathbf{b} = \mathbf{k} \times \mathbf{a}\,, \tag{38}$$

and $\mathbf{k}$ normal to the orbital plane (Eq. 4). We indicate with $\partial/\partial\tau$ the derivative referred to the intermediate frame. The idea is to determine ω from the equation:

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}\tau}\,\mathbf{x} + \frac{\mathrm{d}\eta_2}{\mathrm{d}\tau}\,\mathbf{y} = \frac{\mathrm{d}e}{\mathrm{d}\tau}\,\mathbf{a} + e\,\frac{\partial \mathbf{a}}{\partial\tau}\,, \tag{39}$$

which is derived by differentiation of Eq. (32) with the help of the first relation in (38).

By cross-multiplying Eq. (31) by $\mathbf{k}$ and applying the relations (38) we arrive at the equation:

$$\mathbf{B} = h\,\mathbf{v} - \mathbf{j}\,, \tag{40}$$

where by definition:

$$\mathbf{B} = e\,\mathbf{b}\,. \tag{41}$$

We have:

$$\frac{\partial \mathbf{j}}{\partial\tau} = -\mathbf{w}_{io} \times \mathbf{j} = -\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\,\mathbf{i}\,, \tag{42}$$

$$\frac{\partial \mathbf{v}}{\partial\tau} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} - \mathbf{w}_i \times \mathbf{v} = \left(R + \frac{h}{r}\,\omega - \frac{1}{r^2}\right)\mathbf{i} + \left(T - \frac{\mathrm{d}r}{\mathrm{d}\tau}\,\omega\right)\mathbf{j}\,, \tag{43}$$

where $\mathbf{v}$, $\mathbf{w}_{io}$ and $\mathbf{w}_i$ come from Eqs. (6), (36) and (37). In the same way of Deprit ([9]) we introduce the *effective* perturbing force which in our case takes the expression:

$$\mathbf{Q} = h\,\mathbf{F} + T\,r\,\mathbf{v} + \omega\,\mathbf{e}\,.$$

Equation (40) is differentiated with respect to time and use of Eqs. (8), (42) and (43) brings to:

$$\frac{\partial \mathbf{B}}{\partial\tau} = \left[h\,R + \left(\frac{h^2}{r} - 1\right)\omega\right]\mathbf{i} + h\left(T - \frac{\mathrm{d}r}{\mathrm{d}\tau}\,\omega\right)\mathbf{j} + T\,r\,\mathbf{v}\,. \tag{44}$$

After collecting terms in ω, recognizing that:

$$\frac{h^2}{r} - 1 = \mathbf{e}\cdot\mathbf{i}\,, \qquad\qquad h\,\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\mathbf{e}\cdot\mathbf{j}\,,$$

Eq. (44) is transformed into:

$$\frac{\partial \mathbf{B}}{\partial\tau} = \mathbf{Q} - (\mathbf{Q}\cdot\mathbf{k})\,\mathbf{k}\,. \tag{45}$$

It can be readily found from the definition (41) of $\mathbf{B}$ and from Eq. (45) that the angular velocity of the apsidal frame with respect to the intermediate frame is:

$$\mathbf{w}_{ai} = -\frac{1}{e}\left(\frac{\partial \mathbf{B}}{\partial\tau}\cdot\mathbf{a}\right)\mathbf{k} = -\frac{1}{e}\,(\mathbf{Q}\cdot\mathbf{a})\,\mathbf{k}\,.$$

Thus, the unit vectors $\mathbf{a}$ and $\mathbf{b}$ as seen from the intermediate frame obey to the differential equations:

$$\frac{\partial \mathbf{a}}{\partial\tau} = \mathbf{w}_{ai} \times \mathbf{a} = -\frac{1}{e}\,(\mathbf{Q}\cdot\mathbf{a})\,\mathbf{b}\,, \tag{46}$$

$$\frac{\partial \mathbf{b}}{\partial\tau} = \mathbf{w}_{ai} \times \mathbf{b} = \frac{1}{e}\,(\mathbf{Q}\cdot\mathbf{a})\,\mathbf{a}\,. \tag{47}$$

Moreover, differentiation of Eq. (41), employment of Eqs. (45) and (47) and scalar multiplication by $\mathbf{b}$ yield:

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = \mathbf{Q} \cdot \mathbf{b} \,. \tag{48}$$

We are now ready to express Eq. (39) by means of Eqs. (46) and (48) as:

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}\tau}\,\mathbf{x} + \frac{\mathrm{d}\eta_2}{\mathrm{d}\tau}\,\mathbf{y} = \mathbf{Q} \times \mathbf{k} = \frac{T}{h}\left[(h^2 + r)\mathbf{i} + r\,e\,\mathbf{a}\right] - R\,h\,\mathbf{j} - \omega\,e\,\mathbf{b}\,, \tag{49}$$

where we have recourse to Eq. (5) for $\mathbf{F}$ and to the equation of the hodograph for $\mathbf{v}$, which can be straight derived from Eq. (40).

Before providing the desired expression for ω let us just point out that from Eq. (49) the derivatives of η_1 and η_2 with respect to τ can be written in a very compact way:

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}\tau} = \mathbf{Q} \cdot \mathbf{y}\,, \qquad\qquad \frac{\mathrm{d}\eta_2}{\mathrm{d}\tau} = -\mathbf{Q} \cdot \mathbf{x}\,.$$

These equations are analogous to Eqs. (76) and (77) of the paper by Deprit ([9]) for the elements C and S of his formulation.

Equation (49) is projected along $\mathbf{b}$, then by employing Eqs. (25), (26) and (28) and making appear η_1, η_2 and η_3 through Eqs. (18), (21), (23) and (24) we arrive at the final expression for ω:

$$\omega = \frac{\sqrt{\eta_3}}{\rho\,(1 + l)}\left[R\left(l^2 - 2\,\rho - \rho\,l\right) + T\,(\eta_1 \sin u - \eta_2 \cos u)\,(\rho - l)\right], \tag{50}$$

where ρ and l are introduced in Eqs. (19) and (24). We conclude that the perturbed two-body problem may be described through Eqs (25) - (28) along with the differential equations of the intermediate frame:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u} = \frac{\mathrm{d}\tau}{\mathrm{d}u}\,\mathbf{w}_i \times \mathbf{x}\,, \qquad \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}u} = \frac{\mathrm{d}\tau}{\mathrm{d}u}\,\mathbf{w}_i \times \mathbf{y}\,, \qquad \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}u} = \frac{\mathrm{d}\tau}{\mathrm{d}u}\,\mathbf{w}_i \times \mathbf{k}\,,$$

where $\mathbf{w}_i$ is computed by replacing in Eq. (37) the quantities r, h, $\mathbf{i}$, and ω with the expressions found in Eqs. (18), (23), (33) and (50). The angle ν, previously defined in Eq. (35), is determined by:

$$\nu = \mathrm{atan2}\left(\eta_1 \sin u - \eta_2 \cos u,\, l - \frac{\rho}{l}\right) + \mathrm{atan2}\left(\eta_2,\, \eta_1\right). \tag{51}$$

EDROMO METHOD

In the previous sections we developed a formulation of the perturbed two-body problem that is essentially based on the concept of an intermediate reference frame $(\mathbf{x},\, \mathbf{y},\, \mathbf{k})$. This frame shares the axis $\mathbf{k}$ with the orbital frame and is rotated with respect to it of an angle ν which is equal to $u + f - E$, where u is the independent variable, and f and E are the osculating true and eccentric anomalies (Figure 1). Position and velocity can be calculated from the unit vectors of the intermediate frame, the projections of the eccentricity vector along $\mathbf{x}$ and $\mathbf{y}$, and the semi-major axis.

The EDromo method employs an intermediate frame and describes its orientation by means of a unit quaternion. A more general time-transformation than Eq. (12) is adopted:

$$\frac{\mathrm{d}\tau}{\mathrm{d}\varphi} = \frac{r}{\sqrt{-2\,\varepsilon}}\,, \tag{52}$$

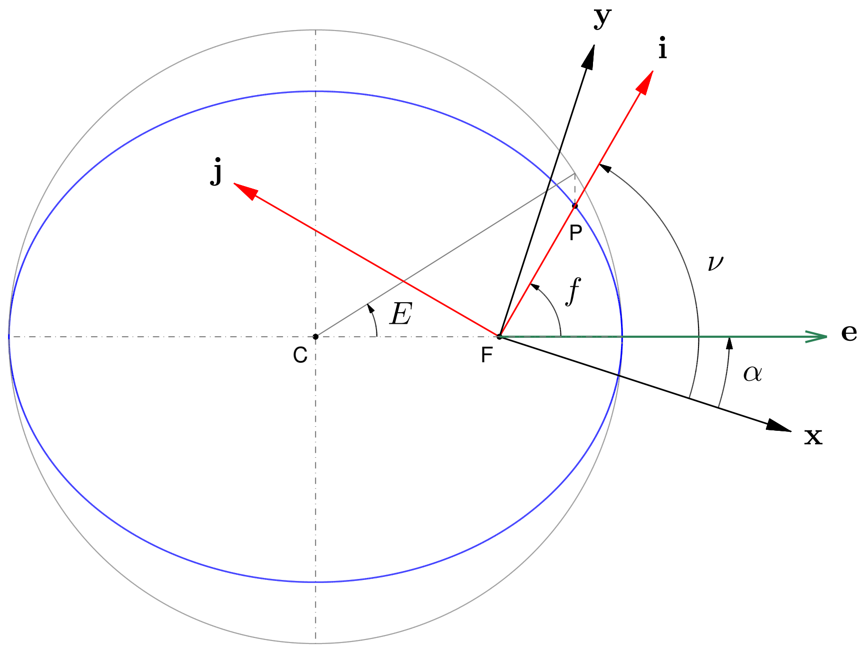

Figure 1. Geometric interpretation of the angles α (Eq. 30) and ν (Eq. 35), which determine the orientation of the axes $(\mathrm{x},\, \mathrm{y})$ of the intermediate frame with respect to the eccentricity vector e and to the axes $(\mathrm{i},\, \mathrm{j})$ of the orbital frame. The point mass in P is moving along an osculating ellipse with center in C and one focus in F, and the angles E and f are the eccentric and true anomalies.

where ε is the total energy:

$$\varepsilon = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} - \frac{1}{r} + \mathcal{U}\,(\tau,\, \mathbf{r})\,, \tag{53}$$

given by the sum of the Keplerian energy and the disturbing potential energy $\mathcal{U}$ which is assumed to depend on time and position. Accordingly we admit that the perturbing force $\mathbf{F}$ in Eq. (2) can be split into two contributions:

$$\mathbf{F} = -\frac{\partial\,\mathcal{U}\,(\tau,\, \mathbf{r})}{\partial\,\mathbf{r}} + \mathbf{P}\,, \tag{54}$$

where $\mathbf{P}$ includes those perturbations that do not arise from a disturbing potential and will be considered in the form:

$$\mathbf{P} = R_p\,\mathbf{i} + T_p\,\mathbf{j} + N_p\,\mathbf{k}\,. \tag{55}$$

First set of elements

Let us first define the quantity λ_3 as:

$$\lambda_3 = -\frac{1}{2\,\varepsilon}\,. \tag{56}$$

The equation of the radial acceleration (11) is converted by operations similar to those applied for the independent variable u into:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} + r - \lambda_3 = \lambda_3\, r\,(r\,R - 2\,\mathcal{U}) + \frac{1}{2\lambda_3}\,\frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi}\,. \tag{57}$$

Equation (53) is differentiated with respect to φ and use of Eqs. (2), (52), (54) and (56) allow to write:

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi} = 2\,\lambda_3^{5/2} r \left(\mathbf{P} \cdot \mathbf{v} + \frac{\partial \mathcal{U}}{\partial \tau} \right), \tag{58}$$

where $\partial/\partial\tau$ indicates the partial derivative with respect to time. In absence of perturbations Eq. (57) reduces to the linear differential equation of an harmonic oscillator of unitary frequency perturbed by the constant λ_3:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} = -r + \lambda_3 \,,$$

which is analogous to Eq. (17). As before we solve this equation to find:

$$r = \lambda_3 \left(1 - \lambda_1 \cos\varphi - \lambda_2 \sin\varphi\right), \tag{59}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \lambda_3 \left(\lambda_1 \sin\varphi - \lambda_2 \cos\varphi\right), \tag{60}$$

where λ_1 and λ_2 are the two constants of integration. Equations (59) and (60) hold also in the perturbed problem but λ_1, λ_2 and λ_3 are in general functions of φ. The method of variation of parameters brings after some algebraic manipulations to the differential equations of λ_1 and λ_2:

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\varphi} = \left(R\,\lambda_3\varrho - 2\mathcal{U}\right)\lambda_3\varrho\,\sin\varphi + \frac{1}{2\lambda_3}\frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi}\left[(1+\varrho)\cos\varphi - \lambda_1\right], \tag{61}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}\varphi} = \left(2\mathcal{U} - R\,\lambda_3\varrho\right)\lambda_3\varrho\,\cos\varphi + \frac{1}{2\lambda_3}\frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi}\left[(1+\varrho)\sin\varphi - \lambda_2\right], \tag{62}$$

where:

$$\varrho = 1 - \lambda_1 \cos\varphi - \lambda_2 \sin\varphi \,. \tag{63}$$

The quantities λ_1, λ_2 and λ_3 are chosen as dependent variables of the EDromo method. Equation (58) still requires some work in order to provide a suitable expression for the angular momentum which appears implicitly through the velocity $\mathbf{v}$. After solving Eqs. (59) and (60) for λ_1 and λ_2 we have:

$$\lambda_1 = (1 + 2\,\varepsilon\, r)\cos\varphi - 2\,\varepsilon\,\frac{\mathrm{d}r}{\mathrm{d}\varphi}\sin\varphi \,, \tag{64}$$

$$\lambda_2 = (1 + 2\,\varepsilon\, r)\sin\varphi + 2\,\varepsilon\,\frac{\mathrm{d}r}{\mathrm{d}\varphi}\cos\varphi \,, \tag{65}$$

where λ_3 has been replaced by the definition in (56). Alternative expressions to Eqs. (64) and (65) are:

$$\lambda_1 = g\cos\left(\varphi - G\right), \tag{66}$$

$$\lambda_2 = g\sin\left(\varphi - G\right), \tag{67}$$

where by means of Eqs. (52), (53) and (6) g takes the form:

$$g = \sqrt{1 + 2\,\varepsilon\left(h^2 + 2\,r^2\,\mathcal{U}\right)}\,, \tag{68}$$

and:

$$G = \operatorname{atan2}\left(-2\,\varepsilon\,\frac{\mathrm{d}r}{\mathrm{d}\varphi},\, 1 + 2\,\varepsilon\, r\right). \tag{69}$$

Note that if $\mathcal{U} = 0$ then by comparing Eqs. (53) and (13) we have $2\,\varepsilon = -1/a$ and therefore g and G simplify into:

$$g = \sqrt{1 - \frac{h^2}{a}} = e\,, \qquad\qquad G = \mathrm{atan2}\left(\frac{\mathrm{d}r}{\mathrm{d}u},\, a - r\right) = E\,,$$

where e and E are the eccentricity and eccentric anomaly. By solving Eq. (68) for the angular momentum h and exploiting Eqs. (56), (59) and (63) we obtain the searched expression of h in terms of λ_1, λ_2, λ_3 and φ:

$$h = \sqrt{\lambda_3\,(m^2 - 2\,\lambda_3\,\varrho^2\,\mathcal{U})}\,, \tag{70}$$

where:

$$m = \sqrt{1 - g^2}\,, \tag{71}$$

and from Eqs. (66) and (67):

$$g = \sqrt{\lambda_1^2 + \lambda_2^2}\,. \tag{72}$$

Finally, Eq. (58) by employing Eqs. (6), (55), (59), (60) and (70) becomes:

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi} = 2\,\lambda_3^3\left[R_p\,(\lambda_1 \sin\varphi - \lambda_2 \cos\varphi) + T_p\,\sqrt{m^2 - 2\,\lambda_3\,\varrho^2\,\mathcal{U}} + \frac{\partial\,\mathcal{U}}{\partial\,\tau}\,\varrho\,\sqrt{\lambda_3}\right]. \tag{73}$$

Equations (61), (62) and (73) along with the time-transformation (52) put in the form:

$$\frac{\mathrm{d}\tau}{\mathrm{d}\varphi} = \lambda_3^{3/2}\,\varrho\,, \tag{74}$$

are the first four differential equations of the EDromo method.

Second set of elements

Let us define on the orbital plane the vector:

$$\mathbf{g} = -\mathbf{i} - \mathbf{c} \times \mathbf{u}\,, \tag{75}$$

where:

$$\mathbf{c} = c\,\mathbf{k}\,, \qquad \mathbf{u} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\,\mathbf{i} + \frac{c}{r}\,\mathbf{j}\,, \qquad c = \sqrt{h^2 + 2\,r^2\,\mathcal{U}}\,. \tag{76}$$

Equation (75) has the same form of Eq. (31) of the *osculating* eccentricity vector $\mathbf{e}$. The magnitude of $\mathbf{g}$ by taking into account Eqs. (76), (53) and (6) is found to be:

$$g = \sqrt{1 + 2\,\varepsilon\,c^2}\,,$$

which is the same expression given in Eq. (68). The angle θ reckoned from $\mathbf{g}$ up to the radial unit vector $\mathbf{i}$ counterclockwise with respect to $\mathbf{k}$ is (see Fig. 2):

$$\theta = \mathrm{atan2}\left(c\,\frac{\mathrm{d}r}{\mathrm{d}\tau},\, \frac{c^2}{r} - 1\right), \tag{77}$$

and it coincides with the true anomaly when the disturbing potential is zero.

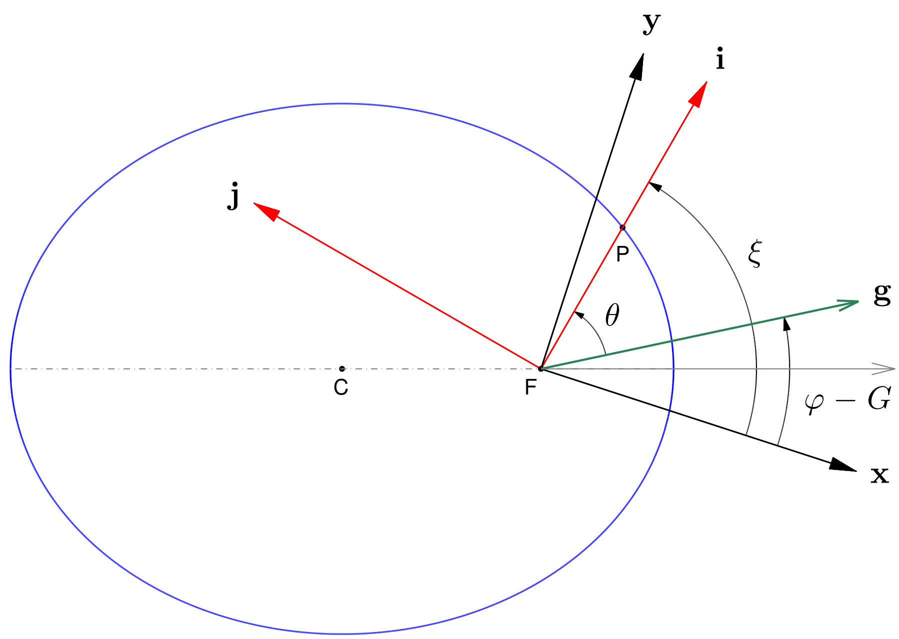

Figure 2. Geometric interpretation of the angles θ (Eq. 77) and ξ (Eq. 79). The generalized eccentricity vector $\mathbf{g}$ does not coincide with the eccentricity vector $\mathbf{e}$ in presence of a disturbing potential energy $\mathcal{U}$ (Eq. 54).

Equations (66) and (67) suggest the existence on the orbital plane of two orthogonal unit vectors $\mathbf{x}$ and $\mathbf{y}$ such that:

$$\mathbf{g} = \lambda_1 \, \mathbf{x} + \lambda_2 \, \mathbf{y} \, . \tag{78}$$

Let the orthonormal basis $(\mathbf{x}, \, \mathbf{y}, \, \mathbf{k})$ be the intermediate frame. Since by definition it is rotated by respect to the orbital frame by the angle (see Fig. 2):

$$\xi = \theta + \varphi - G \, , \tag{79}$$

where G and θ are given in Eqs. (69) and (77) respectively and φ is the independent variable, the angular velocity of this frame results:

$$\mathbf{w}_i = N \, \frac{r}{h} \, \mathbf{i} + \left(\frac{h}{r^2} - \frac{\mathrm{d}\xi}{\mathrm{d}\tau} \right) \mathbf{k} \, . \tag{80}$$

Let us introduce a *unit quaternion* λ that is associated to the intermediate frame:

$$\lambda = \lambda_7 + i \, \lambda_4 + j \, \lambda_5 + k \, \lambda_6 \, ,$$

where i, j, and k are basis elements of the set of quaternions. The components of λ are the four remaining dependent variables of the EDromo formulation. We define also the quaternion:

$$w = i \, q_x + j \, q_y + k \, \Omega \, ,$$

being q_x, q_y and Ω related to the angular velocity $\mathbf{w}_i$ by:

$$q_x = \frac{\mathrm{d}\tau}{\mathrm{d}\varphi}\,(\mathbf{w}_i \cdot \mathbf{x}) = N\,\frac{(\lambda_3\,\varrho)^2}{n}\cos\xi\,, \tag{81}$$

$$q_y = \frac{\mathrm{d}\tau}{\mathrm{d}\varphi}\,(\mathbf{w}_i \cdot \mathbf{y}) = N\,\frac{(\lambda_3\,\varrho)^2}{n}\sin\xi\,, \tag{82}$$

$$\Omega = \frac{\mathrm{d}\tau}{\mathrm{d}\varphi}\,(\mathbf{w}_i \cdot \mathbf{k}) = \frac{n}{\varrho} - \frac{\mathrm{d}\xi}{\mathrm{d}\varphi}\,, \tag{83}$$

where the angular momentum h which enters into the components of $\mathbf{w}_i$ through Eq. (80) is picked from Eq. (70) wherein we operate the replacement:

$$n = \sqrt{m^2 - 2\,\lambda_3\,\varrho^2\,\mathcal{U}}\,, \tag{84}$$

with m reported in Eq. (71). To tackle the computation of $\mathrm{d}\xi/\mathrm{d}\varphi$ in terms of the dependent variables of EDromo we first carry out the cross-product in Eq. (75) and substitute the left-hand side through Eq. (78) to write:

$$\lambda_1\,\mathbf{x} + \lambda_2\,\mathbf{y} = \left(\frac{c^2}{r} - 1\right)\mathbf{i} - c\,\frac{\mathrm{d}r}{\mathrm{d}\tau}\,\mathbf{j}\,.$$

Time-differentiation of the latter equation referred to the intermediate frame produces:

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\,\mathbf{f} = \frac{\mathrm{d}\lambda_1}{\mathrm{d}\tau}\,\mathbf{x} + \frac{\mathrm{d}\lambda_2}{\mathrm{d}\tau}\,\mathbf{y} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{c^2}{r} - 1\right)\mathbf{i} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(c\,\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)\mathbf{j}\,, \tag{85}$$

where we have employed the relations:

$$\frac{\partial\mathbf{i}}{\partial\tau} = \frac{\mathrm{d}\xi}{\mathrm{d}\tau}\,\mathbf{j}\,, \qquad\qquad \frac{\partial\mathbf{j}}{\partial\tau} = -\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\,\mathbf{i}\,,$$

and introduced the vector:

$$\mathbf{f} = \mathbf{k} \times \mathbf{g}\,. \tag{86}$$

By performing the scalar product of both hands of Eq. (85) by $\mathbf{f}$, carrying out the time derivatives on the right-hand side and exploiting Eqs. (59) - (62), (74) and the definition of c in (76) along with Eq. (70) for h one can prove that:

$$\frac{\mathrm{d}\xi}{\mathrm{d}\varphi} = \frac{m}{\varrho} + \frac{1}{m\,(1+m)}\left[(2\mathcal{U} - R\,\lambda_3\varrho)\,(\varrho - m - 2)\,\lambda_3\varrho + \frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi}\,\frac{\zeta\,(m-\varrho)}{2\,\lambda_3}\right], \tag{87}$$

where:

$$\zeta = \lambda_1 \sin\varphi - \lambda_2\cos\varphi\,. \tag{88}$$

The differential equation of λ is obtained by the product of quaternions:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\varphi} = \frac{1}{2}\,\lambda\,w\,,$$

and by components:

$$\frac{\mathrm{d}\lambda_4}{\mathrm{d}\varphi} = \frac{1}{2}\left(\lambda_7\, q_x - \lambda_6\, q_y + \lambda_5\, \Omega\right), \tag{89}$$

$$\frac{\mathrm{d}\lambda_5}{\mathrm{d}\varphi} = \frac{1}{2}\left(\lambda_6\, q_x + \lambda_7\, q_y - \lambda_4\, \Omega\right), \tag{90}$$

$$\frac{\mathrm{d}\lambda_6}{\mathrm{d}\varphi} = -\frac{1}{2}\left(\lambda_5\, q_x - \lambda_4\, q_y - \lambda_7\, \Omega\right), \tag{91}$$

$$\frac{\mathrm{d}\lambda_7}{\mathrm{d}\varphi} = -\frac{1}{2}\left(\lambda_4\, q_x + \lambda_5\, q_y + \lambda_6\, \Omega\right). \tag{92}$$

The angle ξ, which is required for determining q_x and q_y from Eqs. (81) and (82), is computed by writing Eq. (79) as:

$$\xi = \operatorname{atan2}\left(m\, u,\; m^2 - \varrho\right) + \operatorname{atan2}\left(\lambda_2,\; \lambda_1\right),$$

where θ is provided by Eq. (77) and $\varphi - G$ is deduced from Eqs. (66) and (67).

Time-element

A time-element might be included among the state variables instead of the physical time. Let us write Eq. (74) as:

$$\frac{\mathrm{d}\tau}{\mathrm{d}\varphi} = \lambda_3^{3/2}\left(1 - \lambda_1 \cos\varphi - \lambda_2 \sin\varphi\right). \tag{93}$$

In the case of pure Keplerian motion λ_1, λ_2 and λ_3 are constants and the integration of Eq. (93) yields:

$$\tau = A_0 + \lambda_3^{3/2}\left(\varphi - \lambda_1 \sin\varphi + \lambda_2 \cos\varphi\right), \tag{94}$$

where A_0 is the constant of integration. Let us define the *time-element* by:

$$\lambda_0 = A_0 + \lambda_3^{3/2}\, \varphi\,.$$

We first plug λ_0 into Eq. (94) and rearrange the terms to get:

$$\lambda_0 = \tau + \lambda_3^{3/2}\left(\lambda_1 \sin\varphi - \lambda_2 \cos\varphi\right). \tag{95}$$

Then, assuming that the motion is perturbed we differentiate Eq. (95) with respect to φ. This operation requires Eqs. (61), (62), (73) and (74) and the outcome is:

$$\frac{\mathrm{d}\lambda_0}{\mathrm{d}\varphi} = \lambda_3^{3/2} + \lambda_3^{5/2}\left[\left(R\, \lambda_3 \varrho - 2\mathcal{U}\right)\varrho + 2\,\zeta\,\lambda_3\left(R_p\,\zeta + T_p\, n + \frac{\partial \mathcal{U}}{\partial \tau}\,\varrho\,\sqrt{\lambda_3}\right)\right], \tag{96}$$

where ϱ, n and ζ are given in Eqs. (63), (84) and (88).

DIFFERENTIAL EQUATIONS

The EDromo method transforms the six-dimensional Cartesian state vector into eight variables: the time τ; three elements λ_1, λ_2, λ_3 that bring information about the projections of the position and velocity on the *orbital* frame and the orientation of this frame with respect to an *intermediate* frame; and the components λ_4, λ_5, λ_6, λ_7 of a unit quaternion which describes the attitude of the intermediate frame with respect to a fixed frame. A time-element λ_0 can be employed in the state vector of EDromo in place of τ.

We collect below the differential equations of the EDromo method (Eqs. 61, 62, 73, 89 - 92):

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\varphi} = (R\,\lambda_3\varrho - 2\mathcal{U})\,\lambda_3\varrho\,\sin\varphi + \Lambda_3\left[(1+\varrho)\cos\varphi - \lambda_1\right], \tag{97}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}\varphi} = (2\mathcal{U} - R\,\lambda_3\varrho)\,\lambda_3\varrho\,\cos\varphi + \Lambda_3\left[(1+\varrho)\sin\varphi - \lambda_2\right], \tag{98}$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi} = 2\,\lambda_3^3\left(R_p\,\zeta + T_p\,n + \frac{\partial\mathcal{U}}{\partial\tau}\,\varrho\,\sqrt{\lambda_3}\right), \tag{99}$$

$$\frac{\mathrm{d}\lambda_4}{\mathrm{d}\varphi} = \frac{1}{2}\left(\lambda_7\,q_x - \lambda_6\,q_y + \lambda_5\,\Omega\right), \tag{100}$$

$$\frac{\mathrm{d}\lambda_5}{\mathrm{d}\varphi} = \frac{1}{2}\left(\lambda_6\,q_x + \lambda_7\,q_y - \lambda_4\,\Omega\right), \tag{101}$$

$$\frac{\mathrm{d}\lambda_6}{\mathrm{d}\varphi} = -\frac{1}{2}\left(\lambda_5\,q_x - \lambda_4\,q_y - \lambda_7\,\Omega\right), \tag{102}$$

$$\frac{\mathrm{d}\lambda_7}{\mathrm{d}\varphi} = -\frac{1}{2}\left(\lambda_4\,q_x + \lambda_5\,q_y + \lambda_6\,\Omega\right), \tag{103}$$

together with either (Eq. 74):

$$\frac{\mathrm{d}\tau}{\mathrm{d}\varphi} = \lambda_3^{3/2}\varrho\,, \tag{104}$$

or the differential equation (96) for the time-element:

$$\frac{\mathrm{d}\lambda_0}{\mathrm{d}\varphi} = \lambda_3^{3/2}\left[1 + (R\,\lambda_3\varrho - 2\mathcal{U})\,\lambda_3\varrho + 2\,\Lambda_3\,\zeta\right]. \tag{105}$$

Where:

$$\begin{gathered}
\Lambda_3 = \frac{1}{2\lambda_3}\,\frac{\mathrm{d}\lambda_3}{\mathrm{d}\varphi}\,, \\
\varrho = 1 - \lambda_1\cos\varphi - \lambda_2\sin\varphi\,, \\
\zeta = \lambda_1\sin\varphi - \lambda_2\cos\varphi\,, \\
m = \sqrt{1 - \lambda_1^2 - \lambda_2^2}\,, \\
n = \sqrt{m^2 - 2\,\lambda_3\,\varrho^2\,\mathcal{U}}\,,
\end{gathered} \tag{106}$$

are just auxiliary variables. In addition:

$$\begin{gathered}
q_x = N\,\frac{(\lambda_3\,\varrho)^2}{n}\cos\xi\,, \\
q_y = N\,\frac{(\lambda_3\,\varrho)^2}{n}\sin\xi\,, \\
\xi = \operatorname{atan2}\left(m\,u,\; m^2 - \varrho\right) + \operatorname{atan2}\left(\lambda_2,\; \lambda_1\right), \\
\Omega = \frac{n-m}{\varrho} + \frac{1}{m\,(1+m)}\left[(2\mathcal{U} - R\,\lambda_3\varrho)\,(2 - \varrho + m)\,\lambda_3\varrho + \Lambda_3\,\zeta\,\left(\varrho - m\right)\right],
\end{gathered} \tag{107}$$

and given the perturbing vector:

$$\mathbf{F} = -\frac{\partial\mathcal{U}\,(\tau,\,\mathbf{r})}{\partial\mathbf{r}} + \mathbf{P}\,,$$

we have:

$$R = \mathbf{F}\cdot\mathbf{i}\,, \qquad N = \mathbf{F}\cdot\mathbf{k}\,, \qquad R_p = \mathbf{P}\cdot\mathbf{i}\,, \qquad T_p = \mathbf{P}\cdot\mathbf{j}\,,$$

where **i**, **j**, **k** depend on λ_i, $i = 1,\,\ldots,\,7$, as shown in the next subsection.

Table 1. Examples to test. They differ for the initial osculating eccentricity and the perturbations.

example	eccentricity	perturbation
E1	0.95	J_2
E2	0	J_2 + drag
E3	0.3	J_2 + Moon
E4	0.7	J_2 + Moon
E5	0.95	J_2 + Moon

PERFORMANCE ANALYSIS

In this section we analyze the performance of EDromo, with and without a time-element, for the orbit propagation of a perturbed body around the Earth. As performance metric we consider the computational cost versus the accuracy. In the sequel we briefly describe the examples to test, list the methods included in the comparison, explain the numerical tests and show the results. Examples and tests proposed here are the same as in ref. ([7]), so we refer the reader to the section "Performance of the method" of this paper for more details about the implementation of the perturbing forces and the procedure followed to obtain the performance diagrams.

Examples and formulations

An *example* contains the following informations: the initial position and velocity vectors with respect to the central body, and the applied perturbations.

Table (1) lists the five examples considered in this paper which are labelled by the letter E followed by a number in order to easily refer to them when necessary. Examples E1 and E5 adopt the initial position and velocity reported in Table 1 of ref. ([7]) which were used by Stiefel and Scheifele in several problems contained in Section 23 of their book ([10]). These initial conditions are relative to the perigee of an elliptical orbit with an inclination of 30 degrees and eccentricity of 0.95. In examples E2, E3 and E4 the position and direction of the velocity are the same as in E1 and E5 while the velocity magnitude is fixed by the value of the eccentricity written in Table (1). Since the radius of perigee is unchanged (6800 km) a smaller eccentricity implies a smaller semi-major axis.

The sources of perturbation taken into account are the zonal harmonic J_2 of the geopotential, the Moon's gravitational attraction and the atmospheric drag which are combined as shown in Table (1). The formulae needed to compute the corresponding forces are specified in ref. ([7]).

As concerns the *formulations* we compare the method presented here without and with the time-element, the special perturbation method published by Peláez et al. ([3]), its improved version for taking advantage of perturbations arising from disturbing potentials ([7]), and the sets of elements of Stiefel & Scheifele ([10] Section 19) and of Sperling & Burdet ([11] Chapter 9). They are respectively referred to as EDromo, EDromo(te), Dromo, Dromo(P), StiSche and SpBu.

We advice that StiSche and SpBu employ a time-element which instead was not implemented in these schemes to obtain the results shown in ref. ([7]). We also notice that for any formulation except Dromo the J_2 perturbation is introduced by means of a disturbing potential energy $\mathcal{U}$ (Eq. 54).

Computational cost versus accuracy

We want to assess the error in the position at a desired time elapsed from the initial time and the computational cost related to this error.

The time span of propagation for each example corresponds to a number of revolutions which is about 50.5 for E1, 150 for E2, and 49.5 for E3, E4, E5; the exact values in mean solar days (msd) are given in the graphics caption. The numerical integrator is the explicit Runge-Kutta (4, 5) pair of Dormand and Prince ([12]), also called DP54, as coded inside the ode45 function of the Matlab software where local extrapolation is done. The step-size is controlled by the relative and absolute tolerances, the latter has been set equal to 10^{-13}, while the former is allowed to take decreasing values inside the interval (10^{-6}, 10^{-10}).

Given an example, selected a formulation and set the relative tolerance of the integrator we can run a simulation which has to be stopped at the desired physical time. To this end we exploit an iterative Newton algorithm to determine the value of the independent variable which corresponds to the final physical time for the current propagation. Then, the *position error* is calculated as the magnitude of the vector difference between the *current* and *reference* positions, and the computational cost is measured by the number of evalutations of the right-hand side of the differential equations of motion, that is the *function calls*. The reference positions for the five examples are reported in Table 2 of paper ([7]) and were obtained by comparing several formulations integrated by the DP54 method with the absolute and relative tolerances set equal to 10^{-13} ([7]).

By varying the relative tolerance in the prescribed range for each formulation and each example we produce the curves displayed in Figures (3) - (7) which represent the number of function calls required to reach a certain level of accuracy. We first note the significant benefit which can be generated by the time-element λ_0 combined with the EDromo spatial variables. As expected, when the motion is nearly circular (example E2) the time-element does not bring any further improvement since the physical time itself behaves like a time-element by increasing almost linearly with the independent variable. EDromo and EDromo(te) exhibit a similar performance also in the example E5, in this case because the non-conservative perturbation due to the Moon introduces a strong non-linearity in the evolution of the time-element which behaves like the physical time.

It is seen that with just a few function calls the EDromo(te) method is always the most accurate with the only exception of E2, where Dromo(P) is better. When considering tighter relative tolerances EDromo(te) is still competitive with the very efficient method StiSche, sometimes reaching a higher accuracy (as in E1, E3, E5).

Finally, it is worth to point out the outstanding performance of the EDromo schemes in the challenging problem related to E5, which is the Example 2b at page 122 of the book ([10]). This problem was also exploited for numerical comparisons by other authors, such as in the papers ([13]) and ([3]), and in the book ([11] pp. 179-180).

CONCLUSIONS

In this paper we present a formulation of the perturbed two-body problem which works with negative energies. The starting idea consists in applying to the Burdet-Ferrándiz spatial decomposition a Sundman time-transformation involving the eccentric anomaly instead of the true anomaly. The method relies on the concept of the *intermediate* frame, which is analogous to the *ideal* frame exploited by Deprit ([9]) and by Peláez et al. ([3]) to develop the Dromo method. This frame is fixed in

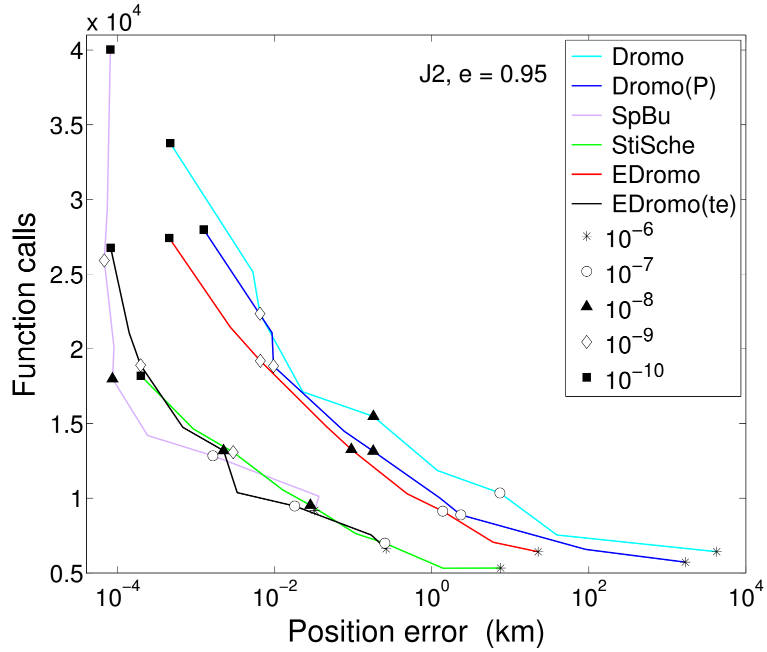

Figure 3. Function calls versus position error for the examples E1. Markers indicate different values of the relative tolerance of the numerical integrator DP54. The time span of propagation is 289.66457509 msd.

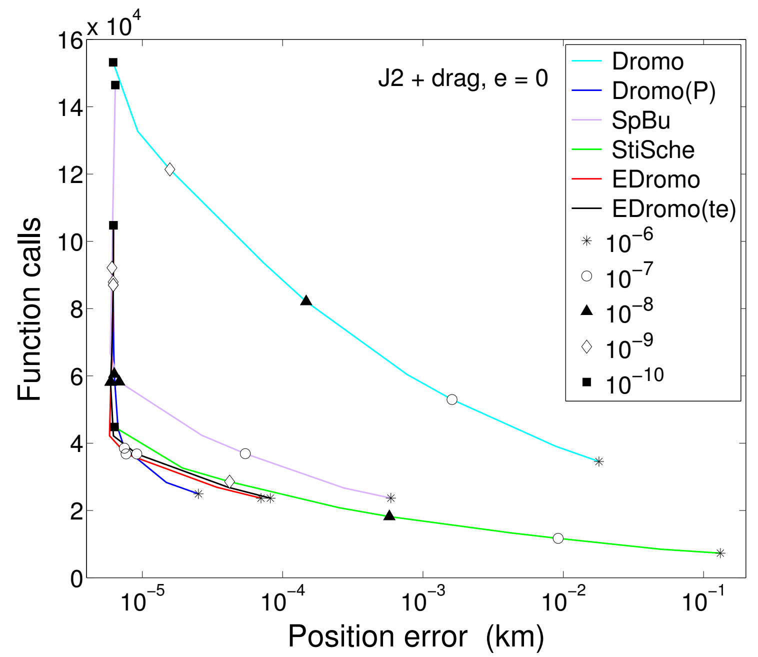

Figure 4. Same as Figure (3) for the example E2. The time span of propagation is 9.68198362 msd.

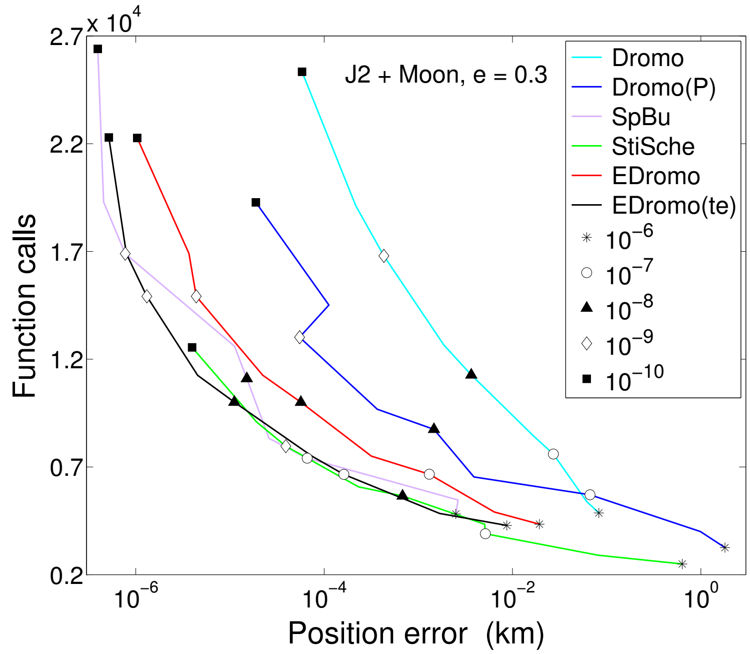

Figure 5. Same as Figure (3) for the example E3. The time span of propagation is 5.45405849 msd.

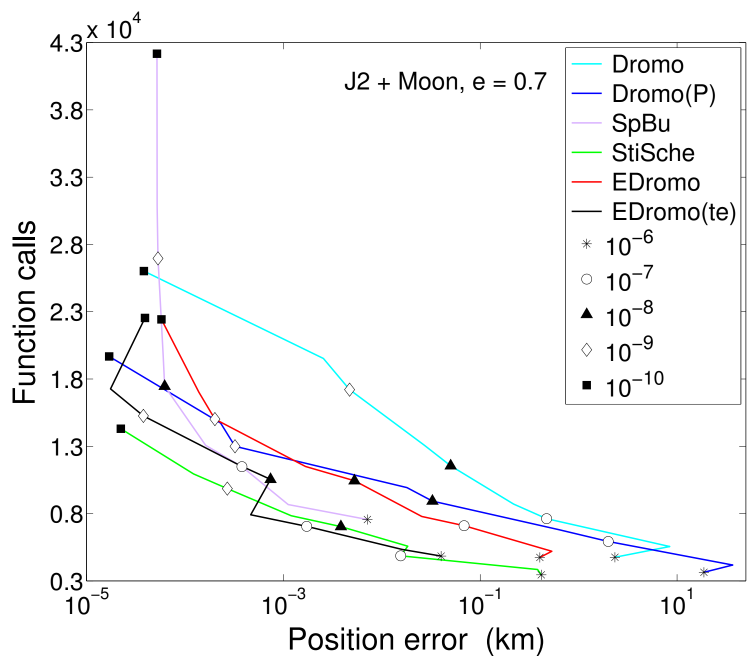

Figure 6. Same as Figure (3) for the example E4. The time span of propagation is 19.43348169 msd.

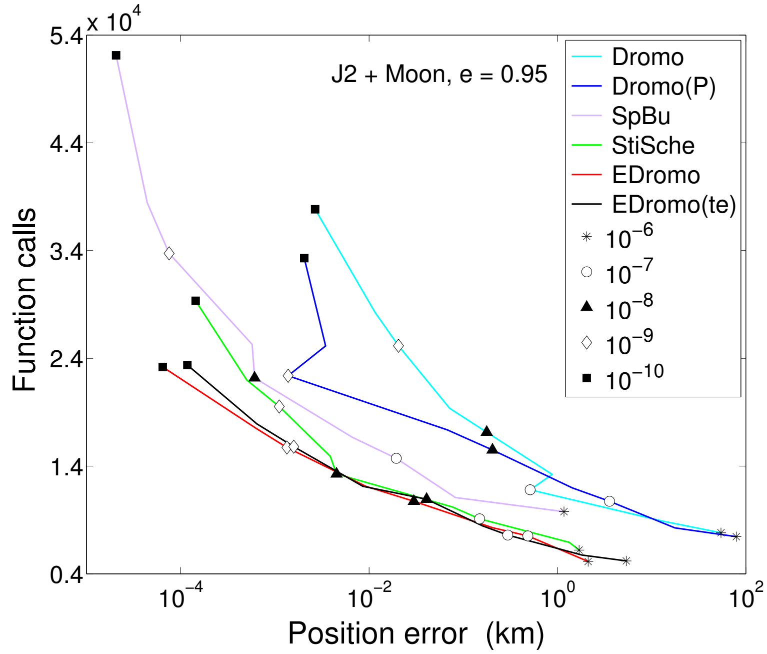

Figure 7. Same as Figure (3) for the example E5. The time span of propagation is 288.12768941 msd.

the pure Keplerian motion and slowly varying for weak and moderate perturbations. By introducing Euler parameters associated to the intermediate frame we can determine the orbital plane orientation in space along with a *reference* direction on this plane, the analogous of the *departure* point in ref. ([9]). The total energy along with two further elements which stem from the dynamics of the orbital radius allow to characterize the relative orientation between the reference and radial directions. The resulting set of seven spatial elements provided by a time-element is named EDromo(te). The proposed method shows a great performance in terms of computational cost versus accuracy, it is always better than the existing formulations of Dromo, both the original ([3]) and the recently improved one ([7]). Moreover the method beats the celebrated set of elements of Stiefel and Scheifele ([10]) for nearly circular and highly eccentric perturbed motion, competing with it in the other cases.

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