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PROPER AVERAGING VIA PARALLAX ELIMINATION

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The elimination of the parallax simplification may deprive the simplified Hamiltonian of the geopotential from some long-period terms of the second order of J_2, thus, making the achievement of a mean elements orbit whose long-period effects are the same as in the osculating orbit unsuccessful. We show how the separation of short- and long-period variations is improved by choosing properly the generating function of the elimination of the parallax transformation.

INTRODUCTION

In a semi-analytical perturbation theory derived from averaging, osculating elements are obtained, first, by propagating the evolution equations and then recovering analytically the short-period effects that were removed by the averaging process. From the mathematical point of view, the computation of the perturbation theory is somewhat arbitrary because secular and long-period terms from which the evolution equations are derived, are *chosen* in this procedure. However, this freedom is limited in engineering applications, where the time history of the *mean elements* may be very useful in itself without the need of recovering the removed short-periodics. Thus, a significant amount of orbit design is done exclusively in mean element space, as is the case, for instance, of satellite constellation design. Also, some geodetic applications, as the determination of time variations of the geopotential harmonics, can be done in mean element space. In these cases, the orbit in mean elements should be "centered"; that is, with no constant or long-periodic offsets from the original, non-averaged orbit.[1,2]

Choosing mean elements which yield centered orbits is not difficult when the mean anomaly appears explicitly in the disturbing function, which happens after the usual expansions in elliptic motion.[3] On the other hand, proceeding in closed-form of eccentricity is highly desirable because of wider generality of a closed-form theory, which, in addition, provides us with a notable economy in the size of the series used. However, the exact separation between short- and long-period variations may be a little tricky when the averaging is computed in closed form. Indeed, the use of standard procedures in the closed-form averaging of the gravitational potential commonly results in a mean elements orbit that misses

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some long-period effects of the second order of J_2, which remain hidden in the generating function of the averaging. To our knowledge, this fact was first noted by Kozai,[4] who recommended adding an integration "constant" —a new term which is independent of the mean anomaly— to the closed form generating function, in such a way that the amended generating function averages to zero in the mean anomaly (see, also Refs. $1, 2, 5$). Nevertheless, the authoritative work of Brouwer[6] seemed to prevail over any other consideration, and Kozai, concerned with finding agreement with Brouwer's expressions, did not include terms of this kind in his theory.

Needless to say that the works of Brouwer and Kozai are correct: Since both deal with the computation of analytical theories, a case in which long-period effects are further averaged in the computation of the secular terms of the theory, Kozai's term does not make any difference because it depends on the argument of the perigee and, therefore, averages out in the computation of secular terms. Besides, since the averaging is mathematically correct, the time-history of osculating elements is thoroughly recovered when recovering the short-period effects by undoing the transformation from mean to osculating elements.

Neither Deprit nor his coworkers seem to have paid too much attention to this point. Absorbed as they were in implementing and debugging procedures useful for mechanized operations, and in particular the elimination of the parallax simplification,[7] they felt confident when reproducing previous trustworthy results in the literature.[8–11*] In this way Deprit seems to miss his own caveat that "The theory of satellites depends vitally on the fact that the expansion of $\mathcal{H}_1$ does not contain the argument 2ω" [12, p. 175]. As a consequence, recent closed-form expressions in the literature, provided as test cases for people interested in building their own tools for Lie transforms averaging, lack of producing centered orbits (see, for instance, Appendix A of Ref. 13). This fact may turn out into a vicious sequence, in which extensions to the Artificial Satellite Theory (AST) by researchers who took these published expressions as benchmarks for their particular implementations,[14,15] are taken in turn by new researchers as the fail-safe reference for their developments.[16] Again, we remark that there is no trouble in using these intermediate expressions in the computation of osculating elements, but some long-period effects will be missed if using them in the computation of the mean elements alone.

The absence of certain second-order long-period effects from the averaged Hamiltonian in closed form is easily checked for the main problem of AST (which only considers the disturbing effect of the J_2 harmonic), for instance, by expanding it in a power series of the eccentricity and comparing the expansion with equivalent terms in Ref. 12. Still, the origin of these differences is not obvious, because simple checks show that generating functions for these published closed-form expressions indeed vanish when averaged over the mean anomaly, thus being free from hidden long-period effects, a fact that adds some puzzlement to this matter. However, this null-averaging happens only because the normalization process is not started from the original Hamiltonian, but from a modified version obtained

*We quote from Ref. [7, p. 112]: "Computer programs [...] are built by stepwise refinements, each extension closely collating its results with developments obtained by various authors. How else could one guard the computer programs from slipping into error?"

after a non-trivial preprocessing based on the elimination of the parallax transformation[7] — a canonical simplification that "worked wonders for the normalization of the main problem of AST" [13, p. 46].

Herewith we demonstrate that the offsets introduced by the closed-form averaging in published solutions relying in the elimination of the parallax, is precisely an effect of this preparatory simplification. Thus, when the elimination of the parallax is performed as reported in the literature, it introduces hidden long-period effects in the first-order generating function. This fact results in a second-order Hamiltonian in closed form that lacks of some long-period effects, a fact that could worsen the performances of existing software if used for computing long-term orbit predictions that only require mean elements propagation (without recovering short-period effects from the transformation from mean to osculating elements). However, we show that this deficiency is easily remedied by supplementing the first order term of the generating function of the elimination of the parallax with the Kozai constant. Furthermore, in order to better illustrate how the occurrence of long-period terms in the generating function of the elimination of the parallax is avoided, we carry out this canonical simplification in Delaunay variables, contrary to the polar variables used in Deprit's original formulation.[7] Besides, our reformulation of the elimination of the parallax algorithm directly in Delaunay variables has the added advantage of releasing this canonical simplification from the subtle algebra of the so-called parallactic functions.

PARALLAX ELIMINATION WITHOUT HIDDEN LONG-PERIODICS

We limit here to the main problem of the Artificial Satellite Theory, whose Hamiltonian is written

$$\mathcal{H} = \mathcal{H}_{0,0} + \mathcal{H}_{1,0}, \tag{1}$$

with

$$\mathcal{H}_{0,0} = -\frac{\mu}{2a}, \tag{2}$$

$$\mathcal{H}_{1,0} = \frac{\mu}{r}\frac{\alpha^2}{r^2}\frac{1}{2}C_{2,0}\left[1-\frac{3}{2}s^2+\frac{3}{2}s^2\cos(2f+2\omega)\right], \tag{3}$$

where, μ is the Earth's gravitational parameter, α is the Earth's equatorial radius, $C_{2,0} = -J_2$ is the second order zonal harmonic coefficient, r is the radius from the Earth's center of mass

$$r = \frac{a\,\eta}{1+e\cos f}, \tag{4}$$

η is the eccentricity function

$$\eta = \sqrt{1-e^2},$$

f is the true anomaly, $s \equiv \sin i$, and $(a, e, i, \Omega, \omega, M)$ is the usual set of orbital elements —standing for semi-major axis, eccentricity, inclination, Right Ascension of the ascending node, argument of the perigee, and mean anomaly, respectively.

One should be aware that, in the Hamiltonian setting, $\mathcal{H}$ must be a function of canonical variables, whereas orbital elements are not canonical. Therefore, throughout this paper it must be understood that orbital elements are not variables by themselves, but functions of some set of canonical variables. Specifically, except otherwise stated, we work with Delaunay variables (ℓ, g, h, L, G, H) where $\ell = M$, $g = \omega$, $h = \Omega$, $L = \sqrt{\mu\, a}$ is the Delaunay action, $G = L\,\eta$ is the modulus of the angular momentum vector, and $H = G\cos i$ is the projection of the angular momentum vector in the direction of the Earth's rotation axis.

The elimination of the parallax[7] is a canonical transformation of the Lie type[17] designed for simplifying the gravity potential by partially averaging short-period terms.[18] It has the effect of reducing Eq. (3) to a trigonometric series in the argument of the perigee, whose coefficients still depend on inverse powers of the radius but limited to the case $1/r^2$. Hence, a subsequent Delaunay normalization[19] of the partially averaged Hamiltonian trivially removes all terms depending on the mean anomaly based on the differential relation

$$a^2\,\eta\,\mathrm{d}\ell = r^2\,\mathrm{d}f, \tag{5}$$

which is derived from the preservation of the angular momentum of Keplerian motion. Splitting the averaging into two canonical transformations instead of a single one has the virtue of dramatically reducing the total number of terms in the generators, which fall down to just one fourth of the number of terms required by a single transformation when computing a third-order solution to the main problem.[9]

As originally devised by Deprit,[7] the elimination of the parallax hinges on a proposition which is stated in the context of polar nodal variables and relies on the special algebra of three state functions, namely the semi-latus rectum and the projections of the eccentricity vector in the nodal reference frame, sometimes called "parallactic" functions.[20,21] Subsequent applications of this successful canonical simplification rely also in the polar nodal setting,[8–11,14,22] which may lead to the view that using polar nodal variables is a key fact for achieving this canonical simplification[23] —an opinion that may well be reinforced by the fact that other useful canonical simplification, the elimination of the latitude, also operates in the polar nodal setting.[24–26] However, this common belief is not true at all, and we show that the elimination of the parallax is more naturally posed in Delaunay variables, thus relieving the procedure from the subtleties of the algebra of the parallactic functions, and hence providing a deeper insight in the simplification strategy. Since we depart from the tradition, we feel compelled to give full details in the perturbation approach.

Assumed a perturbation Hamiltonian $\mathcal{H} = \mathcal{H}_{0,0} + \mathcal{P}$ where $\mathcal{H}_{0,0}$ is integrable, and the disturbing function $\mathcal{P} = \sum_{m\geq 1}(\varepsilon^m/m!)\,\mathcal{H}_{m,0}$ is given by its expansion in powers of some small parameter ε, Deprit's perturbation theory[17] enables the computation of a new, simplified Hamiltonian $\mathcal{H}' = \sum_{m\geq 0}(\varepsilon^m/m!)\,\mathcal{H}_{0,m}$ in new variables, as well as the explicit transformation from old to new variables and vice versa.

Deprit's method is based on the solution of the so-called homological equation

$$\mathcal{L}_0(W_m) = \{W_m, \mathcal{H}_{0,0}\} = \widetilde{\mathcal{H}}_{0,m} - \mathcal{H}_{0,m},$$

where $\{-;-\}$ notes the Poisson bracket operator, and:

- $\widetilde{\mathcal{H}}_{0,m}$ encompasses all known terms from previous computations,
- the term $\mathcal{H}_{0,m}$ is chosen at will (commonly averaging short-period terms),
- the term W_m is solved from the resulting partial differential equation.

This scheme bases on the recurrence known as Deprit's triangle

$$\mathcal{H}_{n,q} = \mathcal{H}_{n+1,q-1} + \sum_{0\le m\le n} \binom{n}{m} \{\mathcal{H}_{n-m,q-1}; W_{m+1}\}, \tag{6}$$

where terms W_{m+1} define the generating function

$$W = \sum_{m\ge 0} \frac{\epsilon^m}{m!} W_{m+1} \tag{7}$$

of the transformation from new to old variables, and, in particular, from mean to osculating elements.

Reduction of parallactic terms $(\alpha/r)^m$ with $m > 2$, is based on the identity

$$\frac{1}{r^m} = \frac{1}{r^2}\frac{1}{r^{m-2}} = \frac{1}{r^2}\left(\frac{1+e\cos f}{a\,\eta^2}\right)^{m-2}, \qquad m > 2. \tag{8}$$

which is immediately derived from Eq. (4). Then, using Eq. (8) the term $\mathcal{H}_{1,0}$ in Eq. (3) is rewritten in the form

$$\begin{aligned}\mathcal{H}_{1,0} \;=\; & \frac{\mu}{2a\,\eta^2}\frac{\alpha^2}{r^2}\frac{1}{2}C_{2,0}\Big[(2-3s^2)\,(1+e\cos f) \\ & +\frac{3}{2}s^2\,e\cos(f+2\omega) + 3s^2\cos(2f+2\omega) + \frac{3}{2}s^2\,e\cos(3f+2\omega)\Big].\end{aligned} \tag{9}$$

Standard perturbation approach

The first order of Deprit's perturbation theory is

$$\mathcal{L}_0(W_1) = \{W_1, \mathcal{H}_{0,0}\} = \mathcal{H}_{1,0} - \mathcal{H}_{0,1},$$

where $\mathcal{H}_{0,0}$ only depends on the Delaunay action L, c.f. Eq. (2), and hence

$$\mathcal{L}_0(W_m) = \{W_m, \mathcal{H}_{0,0}\} = n\,\frac{\partial W_m}{\partial \ell},$$

where $n = \sqrt{\mu/a^3} = \mu^2/L^3$ is the orbit mean motion.

Then, at difference from the usual averaging over the *mean* anomaly ℓ, $\mathcal{H}_{0,1}$ is chosen by removing terms in Eq. (9) in which the *true* anomaly appears explicitly. Thus,

$$\mathcal{H}_{0,1} = \frac{\mu}{2a\,\eta^2}\frac{\alpha^2}{r^2}\frac{1}{2}C_{2,0}\,(2-3s^2). \tag{10}$$

Remark that $\mathcal{H}_{0,1} \neq \langle \mathcal{H}_{1,0} \rangle_f$ because it still depends on r, and $r \equiv r(f)$ as is evident in Eq. (4). Therefore, the Hamiltonian Eq. (10) is affected of long-period terms as well as remaining short-period terms.

Note that the new Hamiltonian term $\mathcal{H}_{0,1}$ should be written in new (prime) variables because it is the result of a canonical transformation

$$(\ell, g, h, L, G, H) \longrightarrow (\ell', g', h', L', G', H').$$

However, for simplicity we avoid the prime notation throughout this paper as far as there is not risk of confusion.

Then, W_1 is solved by quadrature, which is carried out in closed form of the eccentricity based on the standard relation in Eq. (5). Namely,

$$W_1 = \frac{1}{n} \int (H_{1,0} - H_{0,1}) \, \mathrm{d}\ell = \frac{1}{n} \int (H_{1,0} - H_{0,1}) \, \frac{r^2}{a^2 \, \eta} \, \mathrm{d}f,$$

which results in

$$\begin{aligned} W_1 \quad &= \quad n \, \alpha^2 \, \frac{C_{2,0}}{8\eta^3} \Big[(4 - 6s^2) \, e \sin f \\ &\quad +3s^2 \, e \sin(f + 2\omega) + 3s^2 \sin(2f + 2\omega) + s^2 \, e \sin(3f + 2\omega) \Big]. \end{aligned} \tag{11}$$

Note that Eq. (9) may be rewritten as

$$\begin{aligned} \mathcal{H}_{1,0} \quad &= \quad \frac{\mu}{2a} \, \frac{\alpha^2}{r^2} \, \frac{1}{2} C_{2,0} \, \frac{1}{\eta^2} \Big[2 - 3s^2 + 3s^2 \cos 2\theta \\ &\quad + \left(2 - \frac{3}{2} s^2\right) \kappa \cos\theta + \frac{3}{2} s^2 \, \kappa \cos 3\theta + \left(2 - \frac{9}{2} s^2\right) \sigma \sin\theta + \frac{3}{2} s^2 \, \sigma \sin 3\theta \Big], \end{aligned} \tag{12}$$

where $\theta = f + \omega$ is the argument of the latitude, and

$$\kappa = e \cos\omega, \qquad \sigma = e \sin\omega$$

are the components of the eccentricity vector in the nodal frame. Therefore, removing f from Eq. (9) has the same effect as removing the argument of the latitude from Eq. (12), in full agreement with the traditional approach.

The second order of Deprit's perturbation theory is

$$\mathcal{L}_0(W_2) = \widetilde{\mathcal{H}}_{2,0} - \mathcal{H}_{0,2}, \tag{13}$$

where

$$\widetilde{\mathcal{H}}_{2,0} = \{\mathcal{H}_{0,1}, W_1\} + \{\mathcal{H}_{1,0}, W_1\}.$$

After computing the Poisson brackets and using Eq. (8) for eliminating parallactic terms $(\alpha/r)^m$ with $m>2$, we obtain

$$\begin{aligned}
\widetilde{\mathcal{H}}_{2,0} &= -\frac{\mu}{2a}\frac{\alpha^2}{a^2}\frac{\alpha^2}{r^2}C_{2,0}^2\frac{1}{\eta^6}\left\{\frac{5}{2}+\frac{3}{4}e^2+\left(4-\frac{27}{4}s^2+\frac{27}{16}s^4\right)e\cos f\right.\\
&\quad+\left(\frac{3}{4}-\frac{3}{4}s^2-\frac{15}{32}s^4\right)e^2\cos 2f-s^2\left[\frac{21}{4}-\frac{3}{4}e^2+\left(\frac{21}{8}-\frac{45}{16}s^2\right)e^2\cos 2\omega\right.\\
&\quad+\left(\frac{77}{8}-\frac{21}{2}s^2\right)e\cos(f+2\omega)+\left(5-\frac{21}{4}s^2+\frac{3}{8}\left(2-s^2\right)e^2\right)\cos(2f+2\omega)\\
&\quad\left.-\left(\frac{15}{8}-3s^2\right)e\cos(3f+2\omega)-\left(\frac{15}{8}-\frac{39}{16}s^2\right)e^2\cos(4f+2\omega)\right]\\
&\quad+s^4\left[\frac{21}{8}-\frac{15}{32}e^2+\frac{15}{64}e^2\cos(2f+4\omega)+\frac{9}{32}e\cos(3f+4\omega)\right.\\
&\quad\left.\left.-\left(\frac{3}{8}-\frac{3}{32}e^2\right)\cos(4f+4\omega)-\frac{15}{32}e\cos(5f+4\omega)-\frac{9}{64}e^2\cos(6f+4\omega)\right]\right\},
\end{aligned}$$

and choose $\mathcal{H}_{0,2}$ like before: by removing the explicit appearance of f in $\widetilde{\mathcal{H}}_{2,0}$. We get

$$\begin{aligned}
\mathcal{H}_{0,2} &= -\frac{\mu}{2a}\frac{\alpha^2}{a^2}\frac{\alpha^2}{r^2}C_{2,0}^2\frac{1}{8\eta^6}\left[\frac{5}{2}-\frac{21}{4}s^2+\frac{21}{8}s^4\right.\\
&\quad\left.+\left(\frac{3}{4}-\frac{3}{4}s^2-\frac{15}{32}s^4\right)e^2-s^2\left(\frac{21}{8}-\frac{45}{16}s^2\right)e^2\cos 2\omega\right].
\end{aligned}\tag{14}$$

Then, Eq. (13), is solved by quadrature to get

$$\begin{aligned}
W_2 &= -n\,\alpha^2\frac{\alpha^2}{a^2}C_{2,0}^2\frac{1}{\eta^7}\left\{\left(2-\frac{27}{8}s^2+\frac{27}{32}s^4\right)e\sin f+\frac{3}{16}\left(1-s^2-\frac{5}{8}s^4\right)e^2\sin 2f\right.\\
&\quad-\left[\left(\frac{77}{16}-\frac{21}{4}\right)e\sin(f+2\omega)+\left(\frac{5}{4}-\frac{21}{16}s^2+\frac{3}{32}(2-s^2)\,e^2\right)\sin(2f+2\omega)\right.\\
&\quad\left.-\left(\frac{5}{16}-\frac{1}{2}s^2\right)e\sin(3f+2\omega)-\left(\frac{15}{64}-\frac{39}{128}s^2\right)e^2\sin(4f+2\omega)\right]s^2\\
&\quad+\left[\frac{15}{256}e^2\sin(2f+4\omega)+\frac{3}{64}e\sin(3f+4\omega)-\frac{3}{256}(4-e^2)\sin(4f+4\omega)\right.\\
&\quad\left.\left.-\frac{3}{64}e\sin(5f+4\omega)-\frac{3}{256}e^2\sin(6f+4\omega)\right]s^4\right\}.
\end{aligned}\tag{15}$$

Again, by rewritting $\widetilde{\mathcal{H}}_{2,0}$ in terms of θ instead of f, we may check that removing f from $\widetilde{\mathcal{H}}_{2,0}$ provides the same expression as when removing θ, in full agreement with the standard elimination of the parallax procedure.

If iterated once more, the homological equation is

$$\mathcal{L}_0(W_3)=\widetilde{\mathcal{H}}_{3,0}-\mathcal{H}_{0,3},\tag{16}$$

where

$$\widetilde{\mathcal{H}}_{3,0} = \{\mathcal{H}_{0,1} + 2\mathcal{H}_{1,0}, W_2\} + \{\mathcal{H}_{1,1} + \mathcal{H}_{2,0} + \mathcal{H}_{0,2}, W_1\},$$

and the term $\mathcal{H}_{1,1} = \mathcal{H}_{0,2} - \{\mathcal{H}_{0,1}, W_1\}$ was previously computed from Deprit's triangle. After removing parallactic factors, and choosing, like before, $\mathcal{H}_{0,3}$ by removing the explicit appearance of the true anomaly from $\widetilde{\mathcal{H}}_{3,0}$, we get

$$\begin{aligned}\mathcal{H}_{0,3} \;=\;& \frac{\mu}{2a}\frac{\alpha^4}{a^4}\frac{\alpha^2}{r^2}C_{2,0}^3\frac{1}{\eta^{10}}\left[\frac{39}{2} - \frac{567}{8}s^2 + \frac{2961}{32}s^4 - \frac{315}{8}s^6 + \left(\frac{87}{8} - \frac{837}{16}s^2\right.\right.\\ &\left.\left. + \frac{6813}{64}s^4 - \frac{8145}{128}s^6\right)e^2 - \left(\frac{9}{8} + \frac{117}{16}s^2 - \frac{2565}{256}s^4\right)s^2e^2\cos 2\omega\right].\end{aligned}$$

Then, the term W_3 is solved from Eq. (16) and we are ready to compute the next order. Proceeding analogously, we find the next Hamiltonian term:

$$\begin{aligned}\mathcal{H}_{0,4} \;=\;& \frac{\mu}{2a}\frac{\alpha^6}{a^6}\frac{\alpha^2}{r^2}C_{2,0}^4\frac{1}{\eta^{14}}\left\{-\frac{501}{2} + \frac{18909}{16}s^2 - \frac{131157}{64}s^4 + \frac{50049}{32}s^6 - \frac{13815}{32}s^8\right.\\ & + \left(-\frac{3633}{16} + \frac{11961}{16}s^2 + \frac{22509}{128}s^4 - \frac{123309}{64}s^6 + \frac{2596275}{2048}s^8\right)e^2\\ & + \left(-\frac{783}{64} + \frac{13905}{128}s^2 - \frac{26541}{64}s^4 + \frac{277425}{512}s^6 - \frac{1781595}{8192}s^8\right)e^4\\ & + \left[\frac{40545}{32} - \frac{300525}{64}s^2 + \frac{2956191}{512}s^4 - \frac{2360115}{1024}s^6\right.\\ & \left. + \left(-\frac{567}{16} + \frac{7533}{128}s^2 + \frac{50409}{1024}s^4 - \frac{136215}{2048}s^6\right)e^2\right]s^2\,e^2\cos 2\omega\\ & \left.\left(\frac{37611}{512} - \frac{10665}{64}s^2 + \frac{384345}{4096}s^4\right)s^4\,e^4\cos 4\omega\right\}.\end{aligned}$$

By comparison with Table 1 of Ref. 8 we check that the simplified Hamiltonian

$$\mathcal{H}' = \mathcal{H}_{0,0} + \mathcal{H}_{0,1} + \frac{1}{2}\mathcal{H}_{0,2} + \frac{1}{6}\mathcal{H}_{0,3} + \frac{1}{24}\mathcal{H}_{0,4} \tag{17}$$

matches term for term the fourth order Hamiltonian of the main problem after the elimination of the parallax.

Remark that the elimination of the parallax in Delaunay variables can be extended to any order of the perturbation method without leaving the algebra of trigonometric functions, as expected from the analogous properties of the original formulation of this canonical simplification in polar variables.

Perturbation approach without hidden long-periodics

In spite of the elimination of the parallax is just a simplification or partial averaging, contrary to the usual averaging of short-period terms, we may be interested in obtaining a simplified, closed form Hamiltonian that retains all long-period effects from the original problem in addition to the remaining short-period effects associated to $1/r^2$. Otherwise, a later normalization of Eq. (17) will result in an averaged Hamiltonian that, while being totally free from short-period effects, lacks of some long-period effects of the original problem. More specifically, we may want to force the generating function of the elimination of the parallax to be free from long-period effects, a fact that is not evident from Eqs. (11) or (15) because, due to the closed form used in the simplification, the generating function of the transformation is made of periodic terms such that all of them depend on true, contrary to mean anomaly.

Therefore, in order to check wether W is free or not from hidden long-periodics we compute its average over the *mean* anomaly. For the first order term in Eq. (11), we find

$$R_1 = \langle W_1 \rangle_\ell = \frac{1}{2\pi} \int_0^{2\pi} W_1 \frac{r^2}{a^2\,\eta}\,\mathrm{d}f = -n\,\alpha^2\,\frac{C_{2,0}}{8\eta^3}\,\frac{1-\eta}{1+\eta}\,(1+2\eta)\,s^2 \sin 2\omega, \tag{18}$$

which is precisely the term originally pointed out by Kozai (last equation of section II of Ref. 4).

As mentioned above, the fact that the generating function term W_1 is affected of long-period terms in addition to short-period terms, has the effect of depriving $\mathcal{H}_{0,2}$ of some long-period terms which are part of the original main problem of AST in Eq. (1). This effect is completely inherited in a following Delaunay normalization,[19] whose Hamilton equations are affected equally, therefore resulting in orbits in mean elements showing different long term-effects that corresponding orbits in osculating elements*. However, since R_1 is "constant" with respect to ℓ, the trouble is easily remedied by taking a new generating function

$$\mathcal{W} = \sum_{m\geq 0} \frac{\epsilon^m}{m!}\,\mathcal{W}_{m+1}, \tag{19}$$

whose first term is $\mathcal{W}_1 = W_1 - R_1$, which obviously averages to zero over the mean anomaly.

In consequence, using the new generating function we get

$$\begin{aligned}\mathcal{H}_{0,2} \;=\; & -\frac{\mu}{2a}\,\frac{\alpha^2}{a^2}\,\frac{\alpha^2}{r^2}\,C_{2,0}^2\,\frac{1}{\eta^6}\left\{\frac{5}{2} - \frac{21}{4}s^2 + \frac{21}{8}s^4 + \frac{3}{4}\left(c^2 - \frac{5}{8}s^4\right)e^2\right. \\ & \left. - \left[\frac{21}{8} - \frac{45}{16}s^2 + \left(3 - \frac{15}{4}s^2\right)\frac{1+2\eta}{(1+\eta)^2}\right]s^2 e^2 \cos 2\omega\right\},\end{aligned} \tag{20}$$

where $c^2 = 1 - s^2 \equiv \cos^2 i$. If required, the computation of the term $\mathcal{W}_2$ of the new generating function is solved analogously from the corresponding homological equation. Again, we need to incorporate to $\mathcal{W}_2$ a remainder R_2 independent of ℓ such that it guarantees the

*It worths mentioning that the closed from generating function of a Delaunay normalization of Eq. (17) is completely free from long-period effects, as may easily checked.

zero averaging of the second order term of the generating function over the mean anomaly $\langle \mathcal{W}_2 \rangle_\ell = 0$.*

Proceeding this way to higher orders, we find that new long-period corrections should be added to corresponding Hamiltonian terms after the elimination of the parallax. In particular, we find new terms in $\cos 2\omega$ and $\cos 4\omega$ in the third order Hamiltonian:

$$\begin{aligned}
\mathcal{H}_{0,3} \;=\;& \frac{\mu}{2a}\frac{\alpha^4}{a^4}\frac{\alpha^2}{r^2}\frac{C_{2,0}^3}{\eta^{10}}\left[\frac{39}{2}-\frac{567}{8}s^2+\frac{2961}{32}s^4-\frac{315}{8}s^6+\left(\frac{87}{8}-\frac{837}{16}s^2\right.\right. \\
&\left.\left.+\frac{6813}{64}s^4-\frac{8145}{128}s^6\right)e^2-\left(\frac{9}{8}+\frac{117}{16}s^2-\frac{2565}{256}s^4\right)e^2s^2\cos 2\omega\right] \\
&-\frac{\mu}{2a}\frac{\alpha^4}{a^4}\frac{\alpha^2}{r^2}\frac{C_{2,0}^3}{\eta^{10}}\frac{9}{64}\frac{1-\eta}{(1+\eta)^2}\left\{\left[4\left(11+44\eta+40\eta^2-30\eta^3-51\eta^4-14\eta^5\right)\right.\right. \\
&-\left(229+916\eta+904\eta^2-424\eta^3-901\eta^4-244\eta^5\right)s^2 \\
&\left.+5\left(40+160\eta+161\eta^2-65\eta^3-149\eta^4-39\eta^5\right)s^4\right]\frac{s^2}{1+\eta} \\
&-\left[8\left(106+282\eta+333\eta^2+198\eta^3+49\eta^4\right)\right. \\
&-\left(2175+5733\eta+6509\eta^2+3591\eta^3+816\eta^4\right)s^2 \\
&\left.+15\left(93+243\eta+265\eta^2+134\eta^3+27\eta^4\right)s^4\right]s^2\cos 2\omega-(1-\eta) \\
&\left.\times\left[75+300\eta+323\eta^2+72\eta^3-\frac{15}{2}(11+44\eta+47\eta^2+10\eta^3)s^2\right]s^4\cos 4\omega\right\}
\end{aligned} \tag{21}$$

LONG-TERM HAMILTONIAN

Once the elimination of the parallax has been performed, we carry otut the standard Delaunay normalization,[19] which transforms $\mathcal{H}(\ell,g,-,L,G,H)\longrightarrow\mathcal{K}(-,g',-,L',G',H')$. The prime notation is again avoided in what follows without risk of confusion.

As the implicit dependence of $\mathcal{H}$ on ℓ comes only from the term $1/r^2$, $\mathcal{H}$ is trivially averaged based on the differential relation given in Eq. (5). Therefore, up to the second order of J_2, we get the long-term Hamiltonian of the main problem in mean elements

$$\begin{aligned}
\mathcal{K} \;=\;& -\frac{\mu}{2a}-\frac{\mu}{2a}\frac{\alpha^2}{a^2}\frac{J_2}{\eta^3}\left(1-\frac{3}{2}s^2\right)-\frac{\mu}{2a}\frac{\alpha^4}{a^4}\frac{J_2^2}{\eta^7}\frac{3}{8}\left\{5-10s^2+\frac{35}{8}s^4+\frac{1}{2}\left(2-3s^2\right)^2\eta\right. \\
&\left.-\left(c^2-\frac{5}{8}s^4\right)\eta^2-\left[\frac{7}{2}-\frac{15}{4}s^2+\left(4-5s^2\right)\frac{1+2\eta}{(1+\eta)^2}\right]s^2e^2\cos 2\omega\right\}
\end{aligned} \tag{22}$$

in agreement with previous expressions in the literature, in which the averaging was (properly) computed without resorting to the elimination of the parallax simplification, cf. [1,

*Note, however, that in spite of introducing Kozai-type terms in orders higher than one in the generating function improves the mean elements propagation, these corrections are not enough to produce actual centered orbits, c.f. Ref. 2 where the use of "filtered" elements, which are obtained after a final non-canonical transformation, is recommended.

p. 103], [27, p. 82].* This agreement is a strong check of the validity of our approach. In addition, the trivial expansion of Eq. (22) in power series of the eccentricity allows for an additional validation of the correctness of the long-term Hamiltonian by comparison with previous results in Ref. 12.

The last term in the curly brackets of Eq. 22 is important in the propagation of mean elements, that is, the numerical integration of Hamilton equations derived from Eq. (22). However, if one is interested only in the secular terms, then terms depending on ω are further averaged, thus arriving at classical results for analytical theories.

For those interested in checking their own implementations, we also provide the third order term of the Hamiltonian after the Delaunay normalization. Namely,

$$\begin{aligned}\mathcal{K}_{0,3} \;=\; & \frac{\mu}{2a}\frac{\alpha^6}{a^6}\frac{C_{2,0}^3}{\eta^{11}(1+\eta)^4}\frac{9}{256}\bigg\{2(1+\eta)\Big[16(1+\eta)^3\left(35+15\eta-9\eta^2-5\eta^3\right) \\ & -8\left(280+945\eta+1039\eta^2+252\eta^3-228\eta^4-117\eta^5-11\eta^6\right)s^2 \\ & +2\left(1820+5925\eta+5529\eta^2-258\eta^3-1998\eta^4-75\eta^5+209\eta^6\right)s^4 \\ & -\left(1925+6090\eta+4799\eta^2-1908\eta^3-2493\eta^4+594\eta^5+465\eta^6\right)s^6\Big] \\ & +(1+\eta)^2\Big[63\left(4-5s^2\right)\left(8-11s^2\right)+6\left(552-1444s^2+945s^4\right)\eta \\ & +\left(2368-5036s^2+2625s^4\right)\eta^2+48\left(21-38s^2+15s^4\right)\eta^3\Big]e^2s^2\cos 2\omega \\ & +2\left[15\left(10-11s^2\right)(1+4\eta)+\left(646-705s^2\right)\eta^2+6\left(24-25s^2\right)\eta^3\right] \\ & \times e^4s^4\cos 4\omega\bigg\}. \end{aligned} \tag{23}$$

Finally, in order to link with the classical formulation of the elimination of the parallax, we show in the appendix how the Kozai term is obtained when using the original procedure in polar variables.

EXAMPLES

We illustrate the differences in the mean elements propagation when using centered and non-centered elements with two examples. The first corresponds to a Molnya-type orbit, with initial mean elements

$$a_0 = 26,562\,\mathrm{km},\quad e_0 = 0.74105,\quad i_0 = 63.43^\circ,\quad \Omega_0 = 0,\quad \omega_0 = 270^\circ,\quad M_0 = 0, \tag{24}$$

and is presented in Fig. 1. Due to the frozen orbit condition in which Molnya orbits are placed, e, i, and ω remain almost constant in both theories, showing a slight departure from each other in the time histories of the centered (red lines) and non-centered theories (blue

*An equivalent expression to Eq. 22 appears also in [28, p. 834], but, apparently, it was borrowed from Ref. 1. Until our knowledge, subsequents papers of Deprit and coworkers dealing with closed form normalization via the elimination of the parallax never tackle the problem of computing closed-form averaged Hamiltonians retaining all long-period effects.

lines). After a 9-years propagation, differences in eccentricity are of the order of 10^{-7}, of hundredths of arc seconds (as) for inclination, and hundredths of degree for the argument of the perigee. Evolution curves for Ω and M overlay at the precision of the graphics and, therefore, Fig. 1 presents only the differences $\Delta\Omega$ and ΔM between centered and non-centered calculations for these mean elements.

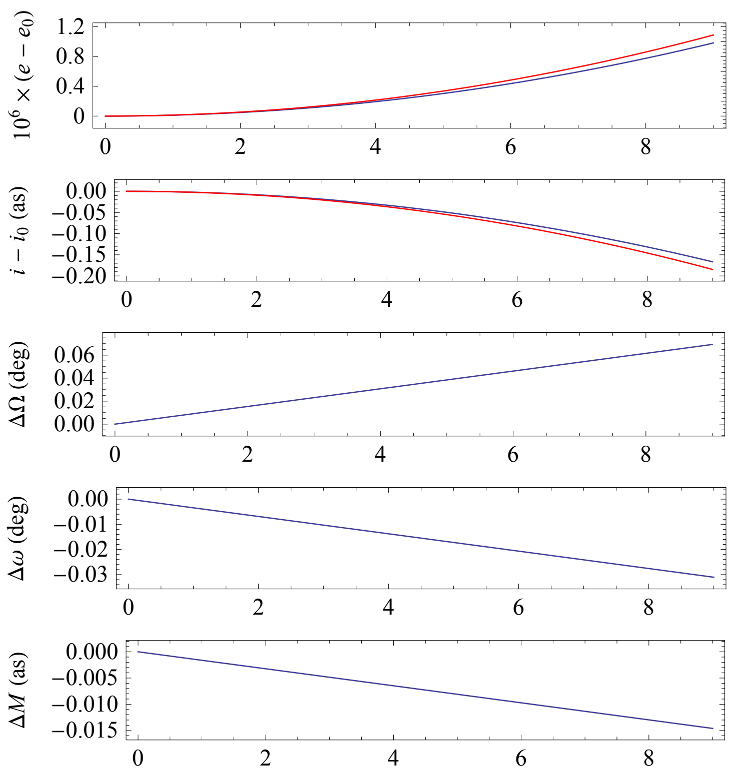

Figure 1. Orbit evolution of a Molnya orbit under the centered (red lines) and uncentered (blue lines) theories assumptions. Abscissas are years. Initial elements are given in Eq. (24).

The other example is a fictitious orbit obtained "unfreezing" the Molnya orbit by reducing the nominal inclination by 10 degrees. Namely,

$$a_0 = 26,562\,\text{km}, \quad e_0 = 0.74105, \quad i_0 = 53.43^\circ, \quad \Omega_0 = 0, \quad \omega_0 = 270^\circ, \quad M_0 = 0, \tag{25}$$

Results of both, centered and uncentered, mean elements propagations are presented in Fig. 2. Now, differences between both theories grow to the order of 10^{-5} for the eccentricity, and several arc seconds for the inclination, whereas differences in the other mean elements remain very similar to the previous case.

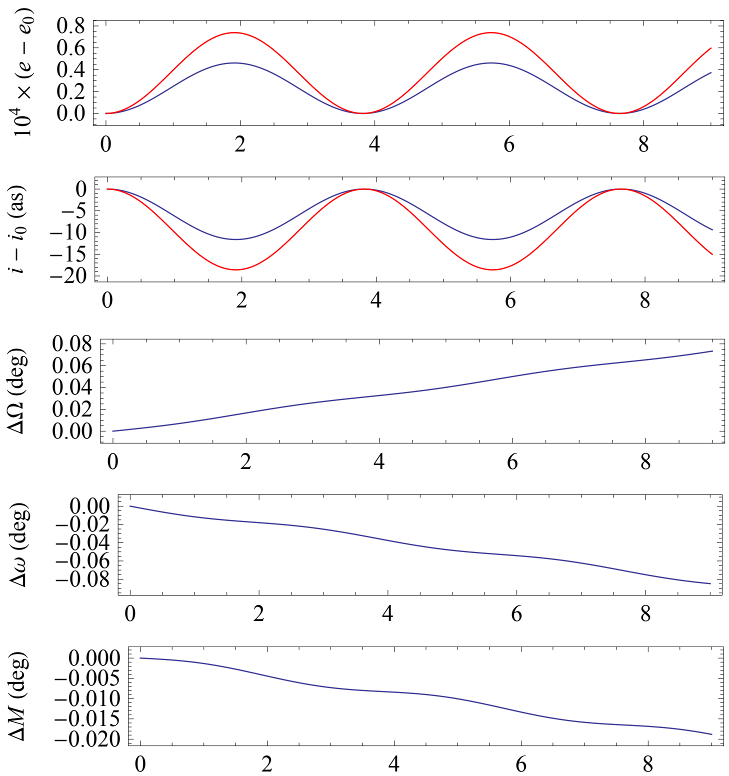

Figure 2. Orbit evolution of the "unfreezed" Molnya orbit with initial elements in Eq. (25). Red lines correspond to the centered theory and blue lines to the uncentered one. Abscissas are years.

CONCLUSIONS

Closed-form, Lie transforms-based analytical and semi-analytical theories running in different laboratories are without any doubt correct. However, closed-form semi-analytical theories may not attain the exact separation between short-period and long-period variations of orbital motion. While this is not a problem in the computation of osculating elements, it may introduce offsets when dealing with applications that only require the propagation of mean elements. This fact is very well-known, and the separation of short- and long-period variations is improved by the simple expedient of adding the Kozai term to the generating function of the Delaunay normalization. However, the use of the elimination of the parallax as a preparatory simplification seems to obscure this fact, because, after the elimination of the parallax has been applied, the subsequent averaging over the mean anomaly no longer needs the addition of Kozai-type terms to guarantee that the generating function of the Delaunay normalization is free from long-periodics. As demonstrated here, in this case Kozai-type terms must be added to the generating function of the elimination of the parallax in order to guarantee its total relief from hidden long-period terms. Proceeding in this way, one easily gets a closed-form Hamiltonian in mean elements retaining all the secular and long-period effects of the original dynamics.

Besides, the reformulation of the elimination of the parallax in Delaunay variables relieves this simplification algorithm from unnecessary subtleties related to the particular algebra of the three parallactic functions in which the original formulation relies upon, thus providing a deeper insight in this Lie transforms simplification technique.

APPENDIX: PROPER PARALLAX ELIMINATION IN POLAR VARIABLES

In the original formulation of the elimination of the parallax Deprit makes use of the state functions

$$\begin{aligned} p &= \frac{\Theta^2}{\mu}, \\ C &= \frac{\Theta}{p}\left(\frac{p}{r}-1\right)\cos\theta + R\sin\theta, \\ S &= \frac{\Theta}{p}\left(\frac{p}{r}-1\right)\sin\theta - R\cos\theta, \end{aligned}$$

where $(r, \theta, \nu, R, \Theta, N)$ is the canonical set of polar nodal variables, standing for radius from the origin, argument of the latitude, argument of the node, radial velocity, modulus of the angular momentum vector, and projection of the angular momentum vector on the z-axis, respectively

The identity

$$\frac{1}{r} = \frac{1}{p}\left(1 + \frac{p\,C}{\Theta}\cos\theta + \frac{p\,S}{\Theta}\sin\theta\right), \tag{26}$$

is used to express inverse powers of the radius as Fourier series in the argument of latitude whose coefficients are functions of p, C, and S. Then, short-period terms are averaged up to a factor of $1/r^2$ by removing the explicit dependence of θ. Since full details on this canonical simplification can be found in the original paper of Deprit,[7] we only present here the expressions that are obtained in the process of computing the long-period Hamiltonian of the main problem.

The main problem in polar nodal variables is

$$\mathcal{M} = \sum_{j\geq 0} \frac{1}{j}\, M_{j,0}, \tag{27}$$

where

$$\begin{aligned} M_{0,0} &= \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r}, \\ M_{1,0} &= \frac{\mu}{2r}\,\frac{\alpha^2}{r^2}\,C_{2,0}\left(1 - \frac{3}{2}s^2 + \frac{3}{2}s^2\cos 2\theta\right), \\ M_{j,0} &= 0, \quad j > 1, \end{aligned}$$

and $s^2 \equiv \sin^2 i = 1 - N^2/\Theta^2$.

The elimination of the parallax transforms Eq. (27) into the new Hamiltonian $\mathcal{M}' = \sum_{j\geq 0}(1/j)\, M_{0,j}$ in new variables $(r', \theta', \nu', R', \Theta', N')$. For simplicity, in what follows we avoid the prime notation without risk of confusion.

To begin with, Eq. (26) is used to write the first-order perturbation as

$$\begin{aligned} M_{1,0} \;=\; & \frac{\Theta^2}{r^2}\,\frac{\alpha^2}{p^2}\,\frac{C_{2,0}}{2}\Big\{1-\frac{3}{2}s^2+\frac{3}{2}s^2\cos 2\theta \\ & +\frac{Cp}{\Theta}\left[\left(1-\frac{3}{4}s^2\right)\cos\theta+\frac{3}{4}s^2\cos 3\theta\right] \\ & +\frac{Sp}{\Theta}\left[\left(1-\frac{9}{4}s^2\right)\sin\theta+\frac{3}{4}s^2\sin 3\theta\right]\Big\}, \end{aligned}$$

Then, we choose $M_{0,1} = \langle M_{1,0}\rangle_\theta$. Namely,

$$M_{0,1} = \frac{\Theta^2}{r^2}\,\frac{\alpha^2}{p^2}\,\frac{C_{2,0}}{2}\left(1-\frac{3}{2}s^2\right),$$

Finally, the first term of the generating function is computed as $U_1 = (r^2/\Theta)\int(M_{1,0} - M_{0,1})\,\mathrm{d}\theta$, which yields

$$\begin{aligned} U_1 \;=\; & \Theta\,\frac{\alpha^2}{p^2}\,\frac{C_{2,0}}{2}\Big\{\frac{3}{4}s^2\sin 2\theta \\ & +\frac{Cp}{\Theta}\left[\left(1-\frac{3}{4}s^2\right)\sin\theta+\frac{1}{4}s^2\cos 3\theta\right] \\ & -\frac{Sp}{\Theta}\left[\left(1-\frac{9}{4}s^2\right)\cos\theta+\frac{1}{4}s^2\cos 3\theta\right]\Big\}+K, \end{aligned}$$

where $K \equiv K(p, C, S, \Theta, N)$ is an arbitrary function that does not depend on θ. The common selection $K = 0$ makes $\langle U_1\rangle_\theta = 0$. However, it must be noted that the argument of latitude comprises short-period effects dependent on the mean anomaly M, as well as long-period effects dependent on the argument of the perigee ω. Hence selecting $K = 0$ introduces hidden long-period terms in U_1.

Indeed, it is easily seen that

$$\frac{1}{2\pi}\int_0^{2\pi} U_1\mathrm{d}\ell = -\Theta\,\frac{\alpha^2}{p^2}\,\frac{C_{2,0}}{2}\,\frac{Cp}{\Theta}\,\frac{Sp}{\Theta}\,\frac{1+2\eta}{(1+\eta)^2}\,s^2+K, \tag{28}$$

where the eccentricity function η in polar nodal variables is

$$\eta = \sqrt{1-(C^2+S^2)\,(p^2/\Theta^2)}.$$

Therefore, to release U_1 from hidden long-period terms dependent on the argument of the perigee, we must choose

$$K = \Theta\,\frac{\alpha^2}{p^2}\,\frac{C_{2,0}}{2}\,\frac{Cp}{\Theta}\,\frac{Sp}{\Theta}\,\frac{1+2\eta}{(1+\eta)^2}\,s^2, \tag{29}$$

which, as expected, is the Kozai constant $-R_1$ given in Eq. (18).

When taking the new value of U_1 in the homological equation, and choosing $M_{0,2}$ by averaging known terms from the Lie triangle over θ, we find

$$
\begin{aligned}
M_{0,2} \;=\;& 2!\frac{\Theta^2}{r^2}\,\frac{\alpha^4}{p^4}\,C_{2,0}^2\bigg[-\frac{5}{8}+\frac{21}{16}s^2-\frac{21}{32}s^4\\
&+\frac{p^2}{\Theta^2}\,C^2\left(-\frac{3}{16}+\frac{27}{32}s^2-\frac{75}{128}s^4\right)\\
&+\frac{p^2}{\Theta^2}\,S^2\left(-\frac{3}{16}-\frac{15}{32}s^2+\frac{105}{128}s^4\right)\\
&+\frac{p^2}{\Theta^2}\,(C^2-S^2)\,\frac{1+2\eta}{(1+\eta)^2}\left(\frac{3}{4}s^2-\frac{15}{16}s^4\right)\bigg],
\end{aligned}
$$

where the last summand in the square brackets is new with respect to analogous expressions provided by Deprit.[7]

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