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CLOSED FORM INTEGRATION OF THE HITZL-BREAKWELL PROBLEM IN ACTION-ANGLE VARIABLES

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As an alternative to recent efforts in giving a complete solution to the attitude propagation of a tumbling triaxial satellite under gravity-gradient, we reformulate the problem in action-angle variables. The new solution is computed by the Lie-Deprit approach and is given in closed form, either for the secular or periodic terms, therefore being valid for any triaxial satellite. Comparisons with other approaches in the literature using nonaction-angle variables show the efficiency of the new solution for a variety of test cases.

INTRODUCTION

Hitzl and Breakwell[1] demonstrated that the problem of the attitude propagation of a triaxial satellite under gravity-gradient perturbations, hereafter the Hitzl-Breakwell problem, admits a closed form solution at least for the long-term changes in the rotational motion of a tumbling satellite. The original solution was based on a perturbation approach in which the torque-free rotation Hamiltonian is taken as the zero-order part. By solving the Hamilton-Jacobi equation, Hitzl and Breakwell find a new set of canonical variables that reduce the free rigid-body Hamiltonian to a quadratic form in the momenta. Incidentally, this quadratic form is formally equivalent, in the new variables, to the secular terms of the rigid body Hamiltonian when expressed in Andoyer variables.

The formulation of the gravity-gradient perturbation in the new set of canonical variables requires the use of Jacobi elliptic functions. Nevertheless, based on the properties of these special functions, Hitzl and Breakwell show that the secular terms of the perturbed problem can be obtained without need of resorting to the usual series expansions of the Jacobi theta functions, thus giving a closed form solution valid for any triaxial body.

Hitzl and Breakwell's solution only deals with the secular terms of the problem, but it has being recently supplied with the periodic terms of the solution, which are also provided in closed form albeit using a slightly different set of canonical variables.[2] In the new solution, the averaging is obtained as the result of a canonical transformation of the Lie type which is computed using Deprit's algorithm.[3] This alternative to the Hitzl and Breakwell's solution is based on a new set of variables that are claimed to reach the complete reduction of the Euler-Poinsot Hamiltonian using a simpler transformation to Andoyer variables. In fact, it

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has been recently argued that all the variables that provide the complete reduction of the Euler-Poinsot Hamiltonian in Andoyer variables may pertain to a unique family.[4]

Neither the Hitzl and Breakwell's set of variables nor the new one are action-angle variables.[5] But, both sets of canonical variables relate to Andoyer variables through explicit transformations, contrary to the case of action-angles, thus simplifying not only the transformation but also its application in a perturbation approach.

Nevertheless, it has been suggested that the use of non-action-angle variables may be related to the introduction of mixed secular-periodic terms in the transformation equations of the averaging,[6] thus constraining the time validity of the solution. This fact compels us to revisit the Hitzl-Breakwell problem from the point of view of action-angle variables and perturbation theory. Thus, we provide a complete closed from solution to the Hitzl-Breakwell problem in action-angle variables, both for the secular and periodic terms, and compare this new solution with previous results[2] showing the pros and cons of each approach.

ZERO ORDER HAMILTONIAN: ANDOYER AND ACTION-ANGLE VARIABLES

With a view on perturbation problems, where the torque-free motion is commonly the zero order in the perturbation approach, the Euler-Poinsot Hamiltonian is formulated in Andoyer variables.[7] Thus, calling $A \leq B \leq C$ to the rigid body's principal momentum of inertia and taking the body frame as defined by the principal axes of inertia, the Hamiltonian of the torque-free rotation is written[8]

$$\mathcal{H} = \left(\sin^2 \nu/A + \cos^2 \nu/B\right)(M^2 - N^2)/2 + N^2/(2C), \tag{1}$$

where $(\lambda, \mu, \nu, \Lambda, M, N)$ are Andoyer variables: the angle λ defines the position of the node of the plane Π orthogonal to the angular momentum vector with respect to the x-y plane of the inertial frame, its conjugate momentum $\Lambda = M \cos I$, where M is the modulus of the angular momentum and I is the inclination between Π and the x-y plane of the inertial frame. The position of the equatorial plane of the rigid body with respect to Π is determined by the node angle μ, and inclination J, which defines $N = M \cos J$. Finally, the x-axis of the body frame is located on the equatorial plane of the body by means of the angle ν.

This set of variables takes advantage of the symmetry introduced by the preservation of the angular momentum vector to reduce the problem to a one-and-a-half degree of freedom system, in which the uncoupled dynamics of ν and N is conveniently represented by simple contour plots of the Hamiltonian. In the particular case of an oblate body, the Hamiltonian in Andoyer variables does not depend on any angle and, therefore, Andoyer variables are action-angle variables. This fact allows to use Andoyer variables in a perturbation approach in those cases in which the rigid body's triaxiality is small, thus allowing to included the triaxiality part of the Hamiltonian with other perturbations. This approach cannot be used in the general case, where it is more suitable the use of variables that completely reduce the Euler-Poinsot Hamiltonian.

Historically, it was Sadov who first reduced the Euler-Poinsot Hamiltonian using action-angle variables.[5] Later, and apparently unaware of the work of Sadov, Hitzl and Breakwell proposed a different set of (non-action-angle) variables that reduce the Euler-Poinsot Hamiltonian in Andoyer variables to its averaged form.[1] The history continues with Kinoshita who, likewise apparently unaware of the work of Sadov, based on Hitzl and Breakwell's variables as an intermediary step in his derivation of the action-angle variables.[6]

As far as Kinoshita was mainly interested in developing a theory for the rotation of a rigid Earth, he felt satisfied with expanding the transformation from Andoyer to action-angle variables in power series of the triaxiality parameter, thus avoiding to deal in the disturbing function with the elliptic integrals that appear in the transformation from Andoyer to action-angle variables. Kinoshita's series expansions make sense because the Earth's figure only departs slightly from an axisymmetric body, a case in which the triaxiality can be treated as a perturbation. Therefore, without need of relying on action angles, Kinoshita's expansions are alternatively derived directly in Andoyer variables by means of a standard Lie transforms approach based on the perturbed spherical rotor.[9]

All previous approaches, and other that appear in the literature, have been recently claimed to pertain to a general family of transformations based on the Hamilton-Jacobi reduction.[4,10] We recall the procedure here for completeness.

Reduction via Hamilton-Jacobi equation

We look for canonical transformations $\mathcal{T}:(\lambda,\mu,\nu,\Lambda,M,N)\to(\ell,g,h,L,G,H)$ that convert Eq. (1) in a new Hamiltonian Φ that depends only on momenta. Because neither λ nor Λ appear in Eq. (1), we choose $h=\lambda$, $H=\Lambda$ and $\Phi\equiv\Phi(L,G)$. The transformation is derived from a generating function in mixed variables $\mathcal{S}=\mathcal{S}(\mu,\nu,L,G)$ such that

$$(\ell,g,M,N)=\partial S/\partial(L,G,\mu,\nu). \tag{2}$$

Because μ is cyclic in Eq. (1), $\mathcal{S}$ is chosen in separate variables $\mathcal{S}=G\,\mu+W(\nu,L,G)$. Then, from Eq. (1) we form the Hamilton-Jacobi equation

$$\left(\frac{\sin^2\nu}{2A}+\frac{\cos^2\nu}{2B}\right)\left[G^2-\left(\frac{\partial W}{\partial\nu}\right)^2\right]+\frac{1}{2C}\left(\frac{\partial W}{\partial\nu}\right)^2=\Phi, \tag{3}$$

where W can be solved by quadrature: $W=G\int\sqrt{Q}\,\mathrm{d}\nu$, where

$$Q=\frac{\sin^2\nu/A+\cos^2\nu/B-1/\Delta}{\sin^2\nu/A+\cos^2\nu/B-1/C}, \tag{4}$$

and

$$\frac{1}{\Delta}=\frac{2\Phi}{G^2}. \tag{5}$$

The transformation Eqs. (2) can be obtained without need of computing W. Thus,

$$\ell = \frac{1}{G}\frac{\partial\Phi}{\partial L}\mathcal{I}_1, \tag{6}$$

$$g = \mu + \mathcal{I}_2 - \left(\frac{2\Phi}{G^2} - \frac{1}{G}\frac{\partial\Phi}{\partial G}\right)\mathcal{I}_1, \tag{7}$$

$$N = G\sqrt{Q}, \tag{8}$$

$$M = G, \tag{9}$$

where

$$\mathcal{I}_1 = \int_{\nu_0}^{\nu}\frac{1}{\sqrt{Q}}\frac{\partial Q}{\partial(1/\Delta)}\,\mathrm{d}\nu, \qquad \mathcal{I}_2 = \int_{\nu_0}^{\nu}\sqrt{Q}\,\mathrm{d}\nu. \tag{10}$$

As far as $|N| \leq M = G$, Eq. (8) implies that $0 \leq Q \leq 1$. Besides, because this must happen for all ν, we get from Eq. (4) that $B \leq \Delta \leq C$ and hence $\frac{1}{2}G^2/C \leq \Phi \leq \frac{1}{2}G^2/B$. Therefore, following derivations are limited to the case of rotations about the axis of maxima inertia. The case of rotations about the axis of minima inertia is approached analogously by a simple permutation of the principal moments of inertia in Eq. (1) and some rearrangement of the functions involved.

The transformation equations for ℓ and g, Eqs. (6) and (7), depend on the integration of the two quadratures in Eq. (10). However, as far as Φ depends only on the momenta G and L, these quadratures can be solved without need of specifying the formal dependence of Φ on the new momenta, thereby giving rise to a whole family of canonical transformations.[9]

The closed form solution of Eq. (10) relies on well known changes of variables. Thus, introducing the parameter $f > 0$ and the function $0 \leq m \leq 1$

$$f = \frac{C\,(B-A)}{(C-B)\,A}, \qquad m = \frac{(C-\Delta)\,(B-A)}{(C-B)\,(\Delta-A)}, \tag{11}$$

and the auxiliary angle ψ defined by

$$\cos\psi = \frac{\sqrt{1+f}\sin\nu}{\sqrt{1+f\sin^2\nu}}, \qquad \sin\psi = \frac{\cos\nu}{\sqrt{1+f\sin^2\nu}}, \tag{12}$$

we find

$$Q = \frac{f}{f+m}\left(1 - m\sin^2\psi\right), \qquad \frac{\partial Q}{\partial(1/\Delta)} = \frac{C\,B}{C-B}\left(1 - \frac{f}{1+f}\cos^2\psi\right),$$

and

$$\mathrm{d}\nu = -\frac{\sqrt{1+f}}{1+f\sin^2\psi}\,\mathrm{d}\psi.$$

Then, the quadratures in Eq. (10) are solved to give

$$\mathcal{I}_1 = \sqrt{1+f}\sqrt{\frac{f+m}{f}}\frac{A\,C}{C-A}\,\mathrm{F}(\psi|m), \tag{13}$$

$$\mathcal{I}_2 = \sqrt{1+f}\sqrt{\frac{f+m}{f}}\left[\frac{m}{f+m}\,\mathrm{F}(\psi|m) - \Pi(-f,\psi\,|\,m)\right], \tag{14}$$

where $F(\psi|m)$ is the elliptic integral of the first kind of elliptic parameter m and amplitude ψ, and $\Pi(-f,\psi\,|\,m)$ is the elliptic integral of the third kind of elliptic parameter m, amplitude ψ, and characteristic $-f$.

Replacing $\mathcal{I}_1$ and $\mathcal{I}_2$ in Eqs. (6)–(9) we get a *family* of canonical transformations. Specific transformations need of particularizing the dependence of the new Hamiltonian Φ on the new momenta L, and G.

Instead of using the new Hamiltonian for parametrizing the family, we note from Eqs. (5) and (11) that Φ is characterized by the identity

$$\Phi = \frac{G^2}{2A}\left(1 - \frac{C-A}{C}\frac{f}{f+m}\right), \tag{15}$$

which can be taken as a definition by assuming that $m = m(L,G)$ in Eq. (15). Then,

$$\frac{\partial\Phi}{\partial L} = \frac{G^2}{2A}\frac{C-A}{C}\frac{f}{(f+m)^2}\frac{\partial m}{\partial L} \tag{16}$$

$$\frac{\partial\Phi}{\partial G} = \frac{2\Phi}{G} + \frac{G^2}{2A}\frac{C-A}{C}\frac{f}{(f+m)^2}\frac{\partial m}{\partial G} \tag{17}$$

and the family of transformations derived from Eqs. (6)–(9) is written

$$\ell = \frac{\sqrt{f\,(1+f)}}{(f+m)^{3/2}}\frac{G}{2}\frac{\partial m}{\partial L}\,\mathrm{F}(\psi|m) \tag{18}$$

$$g = \mu + \sqrt{\frac{f+m}{f}}\sqrt{1+f}\left[\frac{1}{f+m}\left(m + \frac{f}{f+m}\frac{G}{2}\frac{\partial m}{\partial G}\right)\mathrm{F}(\psi|m) - \Pi(-f,\psi|m)\right] \tag{19}$$

$$N = G\sqrt{\frac{f}{f+m}}\sqrt{1-m\sin^2\psi} \tag{20}$$

$$M = G. \tag{21}$$

Action-angle variables

The transformation to action-angle variables requires that the new canonical variables ℓ and g be angles, which are not in general; in particular we note that neither ℓ nor g are angles in the Hitzl and Breakwell's transformation.

Recall that $\mathrm{F}(0|m) = 0$, $\mathrm{F}(2\pi|m) = 4\mathrm{K}(m)$, $\Pi(-f,0|m) = 0$ and $\Pi(-f,2\pi|m) = 4\Pi(-f|m)$, where $\mathrm{K}(m)$ and $\Pi(-f|m)$ are the complete elliptic integrals of the first and third kind, respectively. Then, the angle condition $\oint \mathrm{d}\alpha = 2\pi$ applied to ℓ and g as given in Eqs. (18)–(19), results in

$$2\pi = -\frac{\sqrt{f\,(1+f)}}{(f+m)^{3/2}}\frac{G}{2}\frac{\partial m}{\partial L}\,4\mathrm{K}(m),$$
$$0 = \frac{1}{f+m}\left(m + \frac{f}{f+m}\frac{G}{2}\frac{\partial m}{\partial G}\right)4\mathrm{K}(m) - 4\Pi(-f|m).$$

That is, we need to solve the partial differential system

$$-\frac{L}{G}\,\frac{\pi}{\mathrm{K}(m)}\,\frac{(f+m)^{3/2}}{\sqrt{f\,(1+f)}} = L\,\frac{\partial m}{\partial L}, \tag{22}$$

$$2\,\frac{f+m}{f}\left[(f+m)\,\frac{\Pi(-f|m)}{\mathrm{K}(m)} - m\right] = G\,\frac{\partial m}{\partial G}, \tag{23}$$

or

$$2\,\frac{f+m}{f}\left[(f+m)\,\frac{\Pi(-f|m)}{\mathrm{K}(m)} - m\right] - \frac{L}{G}\,\frac{\pi}{\mathrm{K}(m)}\,\frac{(f+m)^{3/2}}{\sqrt{f\,(1+f)}} = G\,\frac{\partial m}{\partial G} + L\,\frac{\partial m}{\partial L} \tag{24}$$

Since we only need a particular solution satisfying the partial differential equation, we note that the constraint $m = m(L/G)$ makes null the right member of Eq. (24), leading to the trivial solution

$$\frac{L}{G} = \frac{2}{\pi}\sqrt{1+f}\,\sqrt{\frac{f+m}{f}}\left[\Pi(-f|m) - \frac{m}{f+m}\,\mathrm{K}(m)\right], \tag{25}$$

that gives m as implicit function of L/G.* Therefore, when using action-angle variables the Euler-Poinsot Hamiltonian cannot be expressed as explicit function of the momenta L and G, and on the contrary must remain in the implicit form given in Eq. (15).

In summary, the transformation from Andoyer to action-angle variables starts from the computation of Δ from its definition in Eq. (5). In view of $G = M$ and $\Phi = \mathcal{H}$,

$$\Delta = \frac{M^2}{2\mathcal{H}} = \frac{M^2}{\left(\sin^2\nu/A + \cos^2\nu/B\right)(M^2 - N^2) + N^2/C}. \tag{26}$$

Then, $m = m(\Delta)$ is obtained from Eq. (11), and $\psi = \psi(\nu)$ from Eq. (12). Finally,

$$\ell = -\frac{\pi}{2\mathrm{K}(m)}\,\mathrm{F}(\psi|m), \tag{27}$$

$$g = \mu + \sqrt{1+f}\,\sqrt{\frac{f+m}{f}}\left[\frac{\Pi(-f|m)}{\mathrm{K}(m)}\,\mathrm{F}(\psi|m) - \Pi(-f,\psi|m)\right], \tag{28}$$

$$L = \frac{2M}{\pi}\sqrt{1+f}\,\sqrt{\frac{f+m}{f}}\left[\Pi(-f|m) - \frac{m}{f+m}\,\mathrm{K}(m)\right]. \tag{29}$$

The inverse transformation, from action-angle to Andoyer variables, starts solving Eq. (25) for m. Then ψ is computed by inverting Eq. (27)

$$\psi = \mathrm{am}\left(-(2/\pi)\mathrm{K}(m)\,\ell\,\middle|\,m\right) \tag{30}$$

* Note that, as expected, Eq. (25) matches exactly the action-angle definition $L = \frac{1}{2\pi}\oint N\,\mathrm{d}\nu$ for the new momenta L, which is of straightforward derivation after solving Eq. (1) with $\mathcal{H} = \Phi$ for N.

where am is the Jacobi amplitude function. It follows that ν is computed from the inverse transformation of Eq. (12)

$$\cos\nu = -\frac{\sqrt{1+f}\,\sin\psi}{\sqrt{1+f\,\sin^2\psi}}, \qquad \sin\nu = \frac{\cos\psi}{\sqrt{1+f\,\sin^2\psi}}. \tag{31}$$

Finally,

$$\mu = g - \sqrt{1+f}\,\sqrt{\frac{f+m}{f}}\left[\frac{\Pi(-f|m)}{\mathrm{K}(m)}\,\mathrm{F}(\psi|m) - \Pi(-f,\psi|m)\right], \tag{32}$$

$$N = G\sqrt{\frac{f}{f+m}}\,\sqrt{1-m\sin^2\psi}. \tag{33}$$

THE HITZL-BREAKWELL PROBLEM

Hitzl and Breakwell[1] study the problem of the long-term changes in rotational motion of a general triaxial satellite under gravity-gradient. They assume that the dimensions of the satellite are small when compared with the distance to the origin and, therefore, limit to the case of the MacCullag's potential. Besides, they assume that the tumbling frequency (the mean rate of precession of the body z axis about the rotational angular momentum vector) as well as the polhode frequency (the frequency of circulation of the instantaneous axis of rotation relative to the body) are fast when compared with the orbital rate.

If we further assume that the satellite is tumbling at a rate $\mathrm{d}\mu/\mathrm{d}t$ much faster than the rate $\mathrm{d}\nu/\mathrm{d}t$, then we start from a Hamiltonian in Andoyer variables that has been averaged over the μ angle. Namely,[2]

$$\begin{aligned}\mathcal{H} = \frac{M^2}{2C}\Bigg\{1 + \left(\frac{\sin^2\nu}{A/C} + \frac{\cos^2\nu}{B/C} - 1\right)s_J^2 + \frac{1}{4}\left(\frac{n}{M/C}\right)^2\left[1 - 3s_I^2\sin^2(\lambda-\theta)\right] \\ \times\left[\left(2 - \frac{B}{C} - \frac{A}{C}\right)(1-3c_J^2) - \left(\frac{B}{C} - \frac{A}{C}\right)(3-3c_J^2)\cos 2\nu\right]\Bigg\}\end{aligned} \tag{34}$$

where we call $s_J = \sin J$, $c_J = \cos J = N/M$, $s_I = \sin I$, $c_I = \cos I = \Lambda/M$, and $\theta = \theta_0 + n\,t$ where n is the orbital rate, which is constant because we assumed a circular orbit about the disturbing body. The explicit appearance of the time can be avoided by moving to the rotating frame by introducing the Coriolis term $-n\,H$.

Disturbing function in action-angle variables

The formulation of Eq. (34) in solution variables has been given in Ref. 2 in a general form that is valid for any canonical transformation derived from Eqs. (18)–(21). Thus,

$$\mathcal{K} = \Phi - n\,H + U \tag{35}$$

where Φ is in the implicit form of Eq. (15) and

$$U=\frac{n^2}{8}\left[1-3s_I^2\sin^2\phi\right]\left\{(2C-B-A)\left[1-3\,\frac{f}{f+m}\,\mathrm{dn}^2(u|m)\right]\right.$$
$$\left.+3\,(B-A)\left[1-\frac{f}{f+m}\,\mathrm{dn}^2(u|m)\right]\left[1-2\,\frac{(1+f)\,\mathrm{sn}^2(u|m)}{1+f\,\mathrm{sn}^2(u|m)}\right]\right\} \tag{36}$$

with $\phi=h-\theta$, $u=\mathrm{F}(\psi|m)$ and ψ is defined in Eq. (12) as a function of Andoyer's ν. In the present case of action-angle variables, cf. Eq. (27),

$$u=-(2/\pi)\,\mathrm{K}(m)\,\ell \tag{37}$$

Average over ℓ

A first order solution to the Hitzl-Breakwell problem, Eq. (35), can be computed by the standard Lie transforms procedure.[3] Proceeding this way, it has been demonstrated that the secular terms and the generating function of the Lie transformation can also be computed in a general form valid for any canonical transformation derived from the family Eqs. (18)–(21). Thus, cf. Ref. 2, after averaging Eq. (35) over ℓ we get

$$\langle\mathcal{K}\rangle_\ell=\Phi-n\,H+\langle U\rangle_\ell \tag{38}$$

where

$$\langle U\rangle_\ell=n^2\,\frac{B-A}{4}\left\{\frac{C-A}{B-A}+1-3\,\frac{1+f}{m+f}\left[1+\frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\}(1-3s_I^2\sin^2\phi), \tag{39}$$

with $\mathrm{E}(m)$ is the complete elliptic integral of the second kind. Note that, because the averaging is the result of a canonical transformation of the Lie type, the Hamiltonian is assumed to be expressed in new variables (ℓ',g',h',L',G',H'), although we avoid the prime notation for brevity when there is not risk of confusion.

The generating function of the transformation that averages ℓ from the Hamiltonian is

$$W=-\frac{3}{4}\,\frac{n^2}{G}\,(C-B)\,A\,\sqrt{f\,\frac{1+f}{f+m}}\;\mathrm{Z}(\psi|m)\,(1-3s_I^2\sin^2\phi) \tag{40}$$

where $\mathrm{Z}(\psi|m)$ is the Jacobi zeta function, and ψ is given in Eq. (30).

The transformation equations from prime to original variables are then obtained from the generating function in Eq. (40). Thus for $\xi\in(\ell,g,h,L,G,H)$, the corrections $\Delta\xi=\xi-\xi'$

are $\Delta G = 0$ and

$$\Delta\ell = \chi\,\frac{\pi}{2\mathrm{K}(m)}\left[\mathrm{Z}(\psi|m) - 2(f+m)\,\frac{\mathrm{dZ}(\psi|m)}{\mathrm{d}m}\right](1 - 3s_I^2\sin^2\phi) \tag{41}$$

$$\Delta h = \chi\sqrt{f\,\frac{1+f}{m+f}}\,\mathrm{Z}(\psi|m)\,6c_I\sin^2\phi \tag{42}$$

$$\Delta g = -\chi\sqrt{f\,\frac{1+f}{m+f}}\,\mathrm{Z}(\psi|m)\,(1 - 3s_I^2\sin^2\phi) - \frac{L}{G}\,\Delta\ell - \frac{H}{G}\,\Delta h \tag{43}$$

$$\Delta L = \chi\sqrt{f\,\frac{1+f}{m+f}}\,\frac{2\mathrm{K}(m)}{\pi}\left[\frac{\mathrm{E}(m)}{\mathrm{K}(m)} - \mathrm{dn}(u|m)^2\right](1 - 3s_I^2\sin^2\phi) \tag{44}$$

$$\Delta H = \chi\sqrt{f\,\frac{1+f}{m+f}}\,\mathrm{Z}(\psi|m)\,3s_I^2\sin 2\phi \tag{45}$$

where we abbreviate $\chi = \frac{3}{4}\,(n^2/G^2)\,(C-B)\,A$, and the right member of the equations should be assumed in the prime variables. Taking into account the explicit dependence of ψ on the elliptic parameter, as given in Eq. (30), we find

$$\frac{\mathrm{dZ}(\psi|m)}{\mathrm{d}m} = \mathrm{cn}(u|m)\,\frac{\mathrm{sn}(u|m)\,\mathrm{dn}(u|m) - \mathrm{cn}(u|m)\,\mathrm{Z}(\psi|m)}{2(1-m)} \tag{46}$$

with u defined in Eq. (37).

In view of $\psi = \mathrm{am}\,(u|m)$ and the periodicity of all functions involved in Eq. (46), we find that the derivative of the Jacobi Zeta function with respect to the elliptic modulus is a periodic function of u with period $2\mathrm{K}(m)$ —or a periodic function of ℓ with period π. Hence, contrary to the case of variables derived from quadratic Hamiltonians,[2] the use of action-angle variables prevents the introduction of mixed secular-periodic terms in the transformation equations of ℓ and g.

Complete reduction

A new Lie transform $(\ell', g', \phi', L', G', \Phi') \longrightarrow (\ell'', g'', \phi'', L'', G'', \Phi'')$ such that it removes the angle θ is now computed, again by Lie transforms. After the double averaging we find the secular Hamiltonian

$$\begin{aligned}\mathcal{S} = {} & \frac{G^2}{2A}\left(1 - \frac{C-A}{C}\frac{f}{f+m}\right) - n\,H \\ & -n^2\,\frac{B-A}{4}\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{f+m}\left[1 + \frac{C-B}{B}\,\frac{E(m)}{K(m)}\right]\right\}\left(\frac{1}{2} - \frac{3}{2}\frac{H^2}{G^2}\right)\end{aligned} \tag{47}$$

where L, G, and therefore m, and H are constant.

The generating function of the new canonical transformation is

$$V_1 = -n\,\frac{B-A}{4}\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{m+f}\left[1 + \frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\}s_I^2\sin 2\phi \tag{48}$$

which gives rise to the transformation $\xi' = \xi'' + \delta\xi$, $\xi \in (\ell, g, h, L, G, H)$, with $\delta L = \delta G = 0$ and

$$\begin{aligned}
\delta\ell &= \frac{9}{16}\,\frac{n}{G}\,\frac{(C-B)^2\,A}{B\,C}\,\frac{\pi}{2K(m)}\,\frac{\sqrt{f\,(1+f)}}{m\,\sqrt{f+m}} && (49)\\
&\quad\times\left[\frac{f+m}{1-m}\,\frac{E^2(m)}{K^2(m)} - 2f\,\frac{E(m)}{K(m)} + f + \frac{C+B}{C-B}\,m\right] s_I^2 \sin 2\phi \\
\delta h &= \frac{3}{8}\,\frac{n}{G}\,\frac{H}{G}\,(B-A)\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{m+f}\left[1 + \frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\}\sin 2\phi && (50)\\
\delta g &= -\frac{H}{G}\,\delta\phi - \frac{L}{G}\,\delta\ell && (51)\\
\delta H &= \frac{n}{2}\,\frac{B-A}{4}\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{m+f}\left[1 + \frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\} s_I^2 \cos 2\phi && (52)
\end{aligned}$$

where the right member of the equations must be expressed in the double-prime variables.

The secular frequencies of the motion are obtained from Hamilton equations

$$\begin{aligned}
\frac{\mathrm{d}\ell}{\mathrm{d}t} &= \frac{G^2}{2A}\,\frac{C-A}{C}\,\frac{f}{(f+m)^2}\,\frac{\partial m}{\partial L} - \frac{3n^2}{8}\,(B-A)\left(1 - 3c_I^2\right)\frac{1+f}{(f+m)^2} && (53)\\
&\quad\times\left\{1 + \frac{C-B}{B}\left[\frac{f+m}{2m} - \frac{f}{m}\,\frac{E(m)}{K(m)} + \frac{1}{2m}\,\frac{f+m}{1-m}\,\frac{E^2(m)}{K^2(m)}\right]\right\}\frac{\partial m}{\partial L} \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= H\,\frac{3n^2}{4G^2}(B-A)\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{f+m}\left[1 + \frac{C-B}{B}\,\frac{E(m)}{K(m)}\right]\right\} && (54)\\
\frac{\mathrm{d}g}{\mathrm{d}t} &= \frac{2\Phi}{G} - \frac{H}{G}\,\frac{\mathrm{d}h}{\mathrm{d}t} - \frac{L}{G}\,\frac{\mathrm{d}\ell}{\mathrm{d}t} && (55)
\end{aligned}$$

where the partial derivative of m with respect to the action L is given in Eq. (22).

COMPARISON WITH NON-ACTION-ANGLE VARIABLES

To illustrate the improvements obtained in the periodic terms when using action-angle variables, we provide a sample application. In order to compare with previous results,[2] we base on the orbit and inertia parameters of the extreme case provided by PEGASUS-A satellite, which we take from Ref. 11 except for we consider a Keplerian circular orbit, in agreement with our simplification assumptions. Thus, $A = 1.03068 \times 10^5\,\mathrm{kg\,m^2}$, $B = 3.33455 \times 10^5\,\mathrm{kg\,m^2}$, $C = 3.94992 \times 10^5\,\mathrm{kg\,m^2}$ and $M = 5.842 \times 10^5\,\mathrm{kg\,m^2/min}$, $n = 3.71^\circ/\mathrm{min}$. Besides, we set the initial conditions $\mu = 2\,\mathrm{rad}$, $\nu = 1\,\mathrm{rad}$, $\lambda = -0.1\,\mathrm{rad}$, $J = 10^\circ$, $i = 70^\circ$, in order to compare with previous results in Ref. (2).

For the integration we use internal units such that $M = C = 1$. Then, using Eq. (26)–(29) we transform the initial conditions in Andoyer variables to action-angle variables and assume that $\theta = 0$ at $t = 0$. A new transformation using Eqs. (41)–(45) provides the initial conditions in the single-averaged phase. Finally, using Eqs. (49)–(52) we obtain the

	non-averaged	prime	second prime	secular frequency
ℓ	-0.1479898512	-0.1481370529	-0.1448526999	-0.6501504248
g	1.5775303901	1.5776649618	1.5748527787	1.6830026276
ϕ	-0.1	-0.0999998751	-0.1009172983	-0.0441809428
L	0.9548381630	0.9548769383	0.9548769383	
G	1	1	1	
H	0.3420201433	0.3420169296	0.3531301948	

Table 1. Initial conditions in the different phase spaces (internal units).

initial conditions in the double-averaged phase space, and the secular frequencies are then computed from Eqs. (53)–(55). Corresponding values are given in Table 1.

The propagation of the first order analytical solution is compared with a numerical integration of the non-averaged equations —the Hamilton equations derived from the Hamiltonian (35). That is, we propagate the initial conditions in the third column of Table 1 for several orbital periods using the secular frequencies in the fourth column. Then, we take a sample set of points from the output in the double averaged space and correct them using Eqs. (49)–(52). The new points are corrected again but now using Eqs. (41)–(45).

The errors of the analytical solution are shown in Fig. 1. We note that all the angles are affected of secular errors due to the first order truncation of the theory: about 2 mili-radian times orbital period for ℓ and g, and about half mili-radian times orbital period for h. Besides, all the angles are affected of periodic errors of small amplitude, which are related to half the period of the orbital motion. In what respect to the actions, the amplitude of the periodic error is now more apparent and mask the secular errors. The action L is affected of long and short periodic errors, related to the orbital and rotational motion, respectively, while short periodic errors has a negligible effect in the analytical propagation the action H, which on the contrary is clearly affected by long periodic errors.

In order to illustrate the importance of using action-angle variables, we bring here a similar propagation using a set of variables derived from a quadratic Hamiltonian,[2] which are similar to those used by Hitzl and Breackwell.[1] As pointed out in Ref. 2 the derivative of the Jacobi zeta fundtion is no longer periodic when using these kind of variables, a fact that introduces mixed secular-periodic terms in the transformation equations of the averaging over ℓ. The errors introduced by these mixed terms are evident in the long term propagation of the coordinates ℓ and g, as shown in the first two rows of Fig. 2. On the contrary, neither h nor the momenta variables are affected of mixed errors because the corresponding transformation equations are free from the derivative of the Jacobi zeta function, cf. Eq. (40)–(43) of Ref. 2, therefore, corresponding graphics are not presented. We note that a direct comparison between the analytical solution for the action-angles ℓ and g and analogous variables ℓ and g derived from the reduction of the Euler-Poinsot Hamiltonian in Andoyer variables to a quadratic Hamiltonian is not possible, because of the different nature of each set of variables. Nevertheless, a qualitative similar trend is noted in the secular errors of both types of propagations.

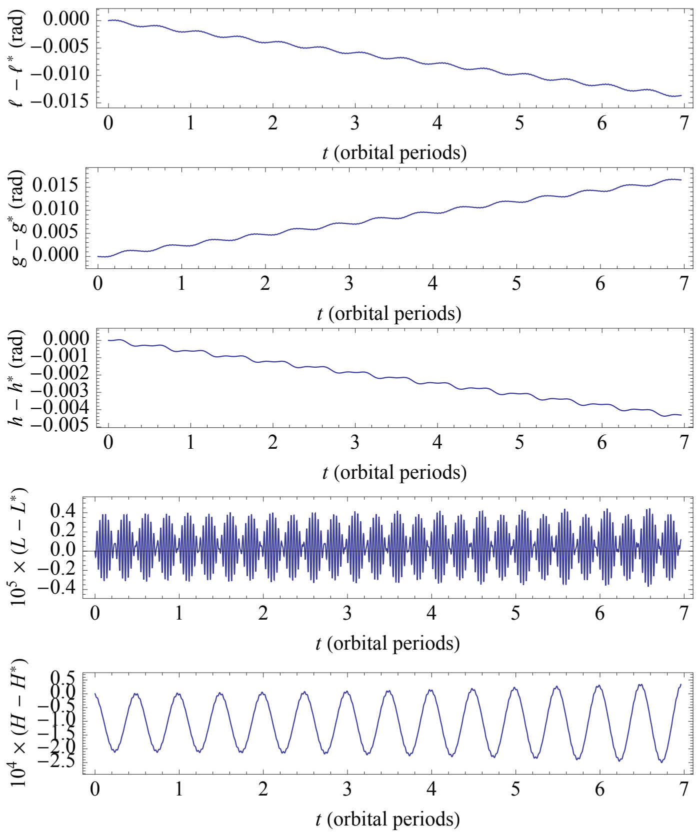

Figure 1. Errors of the action-angle variables propagation (internal units).

CONCLUSIONS

A model has been stablished for the rotation of a triaxial satellite under gravity gradient, in which the orbital frequency is of higher order than the wobble frequency, which in turn is of higher order than the tumbling frequency. This model allows for the analytical investigation of the long-term changes in the rotational motion, but it also provides long-period terms related to the orbital motion, as well as short-period terms related to the precession in inertial space of the body's axis of maximum inertia.

The perturbation solution is based in the use of the action-angle variables that completely reduce the torque-free motion Hamiltonian. When neglecting short-periodic variations related to the tumbling frequency, the disturbing torque is easily cast in these variables and is shown to depend only on Jacobi elliptic functions, although the transformation of Andoyer variables to action-angle variables requires the use of both the incomplete and complete elliptic integrals of the first and third kind.

The solution is computed analytically by means of two consecutive Lie transforms, and is given in closed form of the elliptic parameter up to the first order in the gravity-gradient

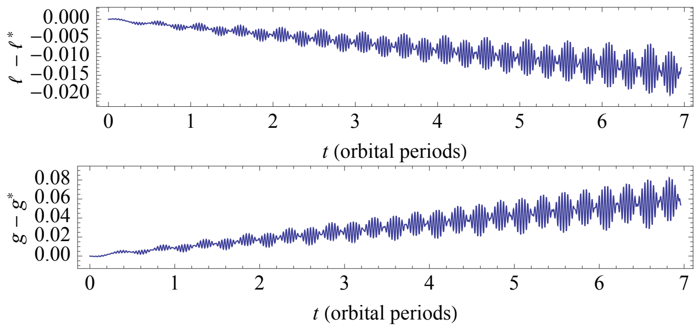

Figure 2. Errors of the propagation using the variables in Ref. 2 (internal units).

perturbation. The Jacobi zeta function plays a prominent role in the computation of the transformation equation that average the short-period terms. Besides, its required derivative with respect to the elliptic modulus is a periodic function when using action-angle variables, thus avoiding the appearance of mixed secular-periodic terms that may appear when using other set of variables.

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