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COMPLETE CLOSED FORM SOLUTION OF A TUMBLING TRIAXIAL SATELLITE UNDER GRAVITY-GRADIENT TORQUE

Martin Lara[*] and Sebastián Ferrer[†]

The attitude dynamics of a tumbling triaxial satellite under gravity-gradient is revisited. The total reduction of the Euler-Poinsot Hamiltonian provides a suitable set of canonical variables that expedites the perturbation approach. Two canonical transformations reduce the perturbed problem to its secular terms. The secular Hamiltonian and the transformation equations of the averaging are computed in closed form of the triaxiality coefficient, thus being valid for any triaxial body. The solution depends on Jacobi elliptic functions and integrals, and applies to non-resonant rotations under the assumption that the tumbling rate is much higher than the orbital or precessional motion.

INTRODUCTION

The rotation of an artificial satellite about its center of mass is described approximately by its torque-free motion. However, because the variety of torques that may act on a satellite,[1] the accuracy of this approximation deteriorates with time, and in spite of the external torques are generally weak, they may induce notable changes in the long-term dynamics. The accurate propagation of the satellite's attitude is commonly approached numerically, although the greater insight on the dynamics provided by approximate analytical solutions has motivated analytical efforts since the early launches of artificial satellites.[2–4]

The analytical approach relies commonly on perturbation strategies in which the zero order is given by the free rigid body motion and other effects are taken as perturbations. Among the variety of perturbations of the torque-free rotation the gravity-gradient is usually the more relevant effect, and the analytical description of the perturbed rotation of an artificial satellite under gravity-gradient torque has attracted a lot of attention in the literature.[5,6] Perturbation schemes are not limited to the approximate analytical integration and have also been successful in the generation of numerical algorithms.[7]

The selection of the variables to represent the motion is crucial to the perturbation approach. We base on a new set of variables that provide complete reduction of the Euler-Poinsot Hamiltonian to a quadratic form,[8] thus making trivial the verification of KAM conditions that guarantee their suitability for applying perturbation methods.

*Grupo de Dinámica Espacial, Universidad de Murcia, Spain, mlara0@gmail.com
†Grupo de Dinámica Espacial, Universidad de Murcia, Spain, sferrer@um.es

The fact that the closed form integration of the Euler-Poinsot problem requires the unavoidable use of elliptic integrals complicates the formulation of the perturbation approach. In some cases, however, the triaxiality coefficient is small and the use of elliptic functions is avoided by the simple expedient of splitting the free rigid body Hamiltonian into an axisymmetric part and a triaxiality "perturbation". The first is taken as the zero order Hamiltonian while the latter is added to the disturbing function, thus the perturbation algorithm can progress smoothly in the realm of trigonometric functions.[9,10] On the contrary, a different approach must be taken when the triaxiality is not small; in that case, it is customary to expand the disturbing function as a Fourier series of Jacobi theta functions.[11] Nevertheless, in some cases this expansion can be avoided and a closed form solution can be achieved at least up to the first order. This is the case of the tumbling satellite which we revisit.[2,4]

In the tumbling satellite approximation, the triaxial body is assumed to be in fast rotation when compared with the orbital rate, and its center of mass is taken in a Keplerian orbit about a spherical perturbing body. Besides, we assume that the dimensions of the satellite are small when compared with its distance to the origin, and hence the disturbing potential is formulated in the MacCullagh approximation. Under these simplifications the secular motion has been derived previously in closed form of the triaxiality coefficient.[2,4] But we can go further and compute, also in closed form, the equations that allow to recover the short periodic terms. All the expressions adopt a compact form in which we need to evaluate both the complete and incomplete forms of the elliptic integrals of the first and the second kind, in addition to the Jacobi elliptic functions. Sample test cases illustrate the performances of the complete closed form solution.

PERTURBATION MODEL AND SIMPLIFICATIONS

Although the attitude of a rigid-body is naturally expressed in Euler angles, the attitude dynamics of artificial satellite may be studied in different representations.[12] Within Hamiltonian formulation, the use of Andoyer variables[13] takes advantage of all the symmetries of the torque-free motion, which can be derived from a Hamiltonian that only depends on an angle, thus revealing its integrable character.[14]

Andoyer variables $(\lambda, \mu, \nu, \Lambda, M, N)$ link the body and inertial frames by means of two sets of Euler-type angles, which relate both frames through an intermediate frame attached to the plane orthogonal to the direction of the angular momentum vector $\boldsymbol{M}$. The position of this plane with respect to the inertial frame is specified by its argument of the node λ and inclination I, which defines $\Lambda = M\cos I$, where $M = \|\boldsymbol{M}\|$. The position of the equatorial plane of the rigid body with respect to the plane perpendicular to the angular momentum is determined by its argument of the node μ and inclination J, which defines $N = M\cos J$. Finally, the x-axis of the body frame is located on the equatorial plane of the body by means of the angle ν.

The Hamiltonian of the torque-free motion in Andoyer variables is[14]

$$\mathcal{H}_0 = \left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)\frac{M^2 - N^2}{2} + \frac{N^2}{2C} = \frac{M^2}{2C}\left[1 + \left(\frac{\sin^2\nu}{A/C} + \frac{\cos^2\nu}{B/C} - 1\right)\sin^2 J\right], \tag{1}$$

where $A \leq B \leq C$ denote the principal moment of inertia, and the body frame is chosen as defined by the principal axes of inertia. We note that λ, Λ and μ are cyclic and therefore $\lambda = \lambda_0$, $\Lambda = \Lambda_0$, and $M = M_0$ are constant. Then the Hamiltonian of the Euler-Poinsot problem is of one degree of freedom, and the integration of the Hamilton equations for μ, ν, and N can be solved by quadrature, accepting a closed form solution in elliptic integrals.

In the case of zero inclination of the angular momentum plane with respect to either the inertial plane or the equatorial plane of the body, Andoyer variables are singular. These singularities are virtual and may be avoided using a different set of variables.[15]

Gravity-gradient torque

To formulate the gravity-gradient torque we make several preliminary assumptions. First, we assume that the dimensions of the rigid body are small when compared with the distance to the perturbing body, which allow us to truncate the disturbing potential to the MacCullagh's term.[16] Besides, we assume that the non-sphericity of the rigid body does not affect its orbital motion about the distant body, which is therefore Keplerian. Finally, we limit to the case of circular orbital motion with constant r.

Under the previous assumptions we may neglect the Keplerian part of MacCullagh's potential and limit our study to the disturbing potential

$$V = -\frac{\mathcal{G}\, m}{2r^3}\,(A + B + C - 3\,D)\,, \tag{2}$$

where $\mathcal{G}$ is the gravitational constant, m is the mass of the disturbing body, r is the distance between the centers of mass of both bodies, and

$$D = A\,\gamma_1^2 + B\,\gamma_2^2 + C\,\gamma_3^2 \tag{3}$$

is the moment of inertia of the rigid body with respect to an axis in the direction of the line joining its center of mass with the perturber, of direction cosines γ_1, γ_2, and γ_3.

Replacing Eq. (3) in Eq. (2), and taking the constraint $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ into account, we get

$$\mathcal{V} = -\frac{M^2}{2C}\left(\frac{n}{M/C}\right)^2\left[\left(1 - \frac{B}{C}\right)\left(1 - 3\gamma_3^2\right) - \left(\frac{B}{C} - \frac{A}{C}\right)\left(1 - 3\gamma_1^2\right)\right], \tag{4}$$

where n is the constant orbital mean motion.

If we choose the orbital plane as the inertial reference frame, then the orbital reference frame is related to the body frame by the composition of the rotations

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = R_3(\nu)\,R_1(J)\,R_3(\mu)\,R_1(I)\,R_3(\phi)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

where $\phi = \lambda - \theta$, and θ is the usual polar coordinate of the orbital motion, which for the assumption of circular motion is $\theta = \theta_0 + n\,t$.

Now, we replace γ_1 and γ_3 as given by Eq. (5) in the disturbing potential Eq. (4) to get:

$$\mathcal{V} = -\frac{M^2}{2C}\left(\frac{n}{M/C}\right)^2 \frac{1}{16}\left[\left(2-\frac{B}{C}-\frac{A}{C}\right)\mathcal{V}_1 + \frac{3}{2}\left(\frac{B}{C}-\frac{A}{C}\right)\mathcal{V}_2\right] \tag{6}$$

that separates the "axisymmetric part" of the potential, which is independent of ν,

$$\begin{aligned}\mathcal{V}_1 \;=\;& (4-6s_J^2)\,(2-3s_I^2+3s_I^2\,C_{2,0,0}) \\ & -12 s_J c_J s_I\,[(1-c_I)\,C_{-2,1,0}+2c_I\,C_{0,1,0}-(1+c_I)\,C_{2,1,0}] \\ & +3s_J^2\,[(1-c_I)^2\,C_{-2,2,0}+2s_I^2\,C_{0,2,0}+(1+c_I)^2\,C_{2,2,0}]\end{aligned} \tag{7}$$

from the "tri-axiality part"

$$\begin{aligned}\mathcal{V}_2 \;=\;& 6s_I^2 s_J^2\,(C_{2,0,-2}+C_{2,0,2}) - 4(1-3c_I^2)s_J^2\,C_{0,0,2} \\ & +(1+c_J)^2\,[(1-c_I)^2\,C_{-2,2,2}+2s_I^2\,C_{0,2,2}+(1+c_I)^2\,C_{2,2,2}] \\ & +(1-c_J)^2\,[(1-c_I)^2\,C_{-2,2,-2}+2s_I^2\,C_{0,2,-2}+(1+c_I)^2\,C_{2,2,-2}] \\ & +4s_I s_J\,(1+c_J)\,[(1-c_I)\,C_{-2,1,2}+2c_I\,C_{0,1,2}-(1+c_I)\,C_{2,1,2}] \\ & -4s_I s_J\,(1-c_J)\,[(1-c_I)\,C_{-2,1,-2}+2c_I\,C_{0,1,-2}-(1+c_I)\,C_{2,1,-2}]\,,\end{aligned} \tag{8}$$

which carries the ν contribution to the perturbation. We abbreviate notation by writing $C_{i,j,k} \equiv \cos(i\phi + j\mu + k\nu)$, and $c_I \equiv \cos I$, $s_I \equiv \sin I$, $c_J \equiv \cos J$ and $s_J \equiv \sin J$, which in Andoyer variables are function only of momenta.

Further simplifications

We further assume that the rate of variation of μ is much faster than the other frequencies of the motion, the rate of variation of ν and the mean orbital motion n. Then, we simplify the disturbing function by neglecting short periodic terms related to μ, to obtain

$$\begin{aligned}\langle\mathcal{V}_1\rangle_\mu &= -2(1-3c_J^2)\left(2-3s_I^2+3s_I^2\cos 2\phi\right) \\ \langle\mathcal{V}_2\rangle_\mu &= 4(1-c_J^2)\left(2-3s_I^2+3s_I^2\cos 2\phi\right)\cos 2\nu.\end{aligned}$$

In our simplifications, the gravity-gradient torque exerted on a tumbling satellite is approximated by

$$\langle\mathcal{V}\rangle_\mu = -\frac{M^2}{2C}\left(\frac{n}{M/C}\right)^2 \frac{1}{16}\left[\left(2-\frac{B}{C}-\frac{A}{C}\right)\langle\mathcal{V}_1\rangle_\mu + \frac{3}{2}\left(\frac{B}{C}-\frac{A}{C}\right)\langle\mathcal{V}_2\rangle_\mu\right] \tag{9}$$

and we call the *tumbling satellite problem* to the system defined by the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \langle\mathcal{V}\rangle_\mu$, that is

$$\begin{aligned}\mathcal{H} \;=\;& \frac{M^2}{2C}\left\{\left(\frac{\sin^2\nu}{A/C}+\frac{\cos^2\nu}{B/C}\right)s_J^2 + c_J^2 + \frac{1}{8}\left(\frac{n}{M/C}\right)^2 (2-3s_I^2+3s_I^2\cos 2\phi)\right. \\ & \left.\times\left[\left(2-\frac{B}{C}-\frac{A}{C}\right)(1-3c_J^2) - \left(\frac{B}{C}-\frac{A}{C}\right)(3-3c_J^2)\cos 2\nu\right]\right\}\end{aligned} \tag{10}$$

where $n^2/(M/C)^2$ is a small quantity and, therefore, the tumbling satellite problem can be approached by perturbation methods.

THE DISTURBING FUNCTION IN SOLUTION VARIABLES

The integration of the perturbed problem Eq. (10) is expedited when using suitable variables such that they reduce the zero-order Hamiltonian, Eq. (1), to a function of only momenta.

Zero order: complete reduction

There are a variety of possibilities in the literature based on the complete reduction of the torque free Hamiltonian in Andoyer variables.[4,17,18] Remarkably, it has been recently pointed out that all of them pertain to the same family of canonical transformations T : $(\lambda,\mu,\nu,\Lambda,M,N)\longrightarrow(\ell,g,h,L,G,H)$.[8] Namely, $h=\lambda$, $H=\Lambda$, and

$$\ell=\frac{1}{2}\frac{\sqrt{f\,(1+f)}}{(f+m)^{3/2}}\frac{\partial m}{\partial\beta}\,\mathrm{F}(\psi|m) \tag{11}$$

$$g=\mu+\sqrt{\frac{1+f}{f}\,(f+m)}\,\left[\frac{1}{f+m}\left(m-\frac{f}{f+m}\,\frac{\beta}{2}\,\frac{\partial m}{\partial\beta}\right)\mathrm{F}(\psi|m)-\Pi(-f;\psi|m)\right] \tag{12}$$

$$N=G\sqrt{\frac{f}{f+m}}\,\sqrt{1-m\sin^2\psi}, \tag{13}$$

$$M=G \tag{14}$$

where $\beta=L/G$, $\mathrm{F}(\psi|m)$ is the incomplete elliptic integral of the first kind of parameter m and amplitude ψ, $\Pi(-f;\psi|m)$ is the incomplete elliptic integral of the third kind of parameter m, amplitude ψ and characteristic $-f$. The value of $f>0$ is given by

$$f=\frac{C\,(B-A)}{(C-B)\,A}, \tag{15}$$

and the elliptic parameter $0\le m\le 1$ by

$$m=\frac{(C-\Delta)\,(B-A)}{(C-B)\,(\Delta-A)}, \tag{16}$$

with $C\ge\Delta=M^2/(2\mathcal{H}_0)\ge B$, which limits the zero-order model to energies in the range $M^2/(2C)\le\mathcal{H}_0\le M^2/(2B)$.* The amplitude ψ is unambiguously defined by

$$\cos\psi=\frac{\sqrt{1+f}\sin\nu}{\sqrt{1+f\sin^2\nu}},\qquad\sin\psi=\frac{\cos\nu}{\sqrt{1+f\sin^2\nu}}. \tag{17}$$

The preceding transformation reduces Eq. (1) to the standard form

$$\mathcal{K}_0=\frac{G^2}{2A}\left(1-\frac{C-A}{C}\frac{f}{f+m}\right), \tag{18}$$

*Because $f>0$, the characteristic of the elliptic integral of the third kind is negative, and hence $\Pi(-f;\psi|m)$ is a *circular case* that can be evaluated using Heuman's Lambda function.[19] This was the choice of Hitzl and Breakwell,[4] as well as Kinoshita.[20]

where $m = m(L, G)$ must be defined for each particular transformation of the general family defined by Eqs. (11)–(14).

Classical selections of the elliptic parameter are

$$m = \frac{(1-\beta^2)\,f}{\beta^2 + f/(2+f)},$$

by Hitzl and Breakwell[4] or, after them, Kinoshita.[20†] In the case of action-angle variables,[17,20] m must remain as an implicit function of $\beta = L/G$. Alternatively, it has been recently claimed that the selection

$$m = f\,\frac{1+f}{\beta^2} - f$$

provides similar features to the case of action-angles but with a higher simplification of the transformation equations, which do not longer rely on the solution of implicit equations for the computation of the elliptic parameter.[8]

We intentionally delay the selection of the elliptic parameter $m = m(\beta) = m(L/G)$ as much as possible. Independently of this choice, if we call

$$u = \mathrm{F}(\psi|m) = \ell/X(m), \qquad X = \frac{1}{2}\frac{\sqrt{f\,(1+f)}}{(f+m)^{3/2}}\,\frac{\partial m}{\partial \beta}, \tag{19}$$

then $\psi = \mathrm{am}(u|m)$ from Eq. (11), where am is the Jacobi amplitude function, and the inverse transformation to Eqs. (11)–(14) is written

$$\cos\nu = \frac{\sqrt{1+f}\,\mathrm{sn}(u|m)}{\sqrt{1+f\,\mathrm{sn}^2(u|m)}}, \qquad \sin\nu = \frac{\mathrm{cn}(u|m)}{\sqrt{1+f\,\mathrm{sn}^2(u|m)}}, \tag{20}$$

where sn and cn are the Jacobi sine amplitude and cosine amplitude elliptic functions, respectively, and

$$\mu = g - \sqrt{\frac{1+f}{f}\,(f+m)}\;[Y(m,\beta)\,u - \Pi(-f;\mathrm{am}(u|m)|m)] \tag{21}$$

$$M = G \tag{22}$$

$$N = G\sqrt{\frac{f}{f+m}}\,\mathrm{dn}(u|m) \tag{23}$$

where dn is the Jacobi delta amplitude elliptic function, and we abbreviate

$$Y = \frac{1}{f+m}\left(m - \frac{f}{f+m}\,\frac{\beta}{2}\,\frac{\partial m}{\partial \beta}\right). \tag{24}$$

†In Hitzl and Breakwell's notation, $f = 2\nu/(1-\nu)$, $\beta^2 = \alpha_{1\,3}^2 = f\,(2+f-m)/((2+f)\,(f+m))$, where the triaxiality parameter $0 \le \nu \le 1$ should not be confused with the Andoyer variable ν used in this paper. Kinoshita changes Hitzl and Breakwell's notation to $e_1 = \nu$ and $b = 1/\alpha_{1\,3}$.

Disturbing function in new variables

It remains to express the disturbing function Eq. (9), and hence c_I, s_J and ν, in the new variables. From Eqs. (22) and (23), we find

$$c_J = \sqrt{\frac{f}{f+m}}\,\mathrm{dn}(u|m),$$

while $c_I = H/G$ because $\Lambda = H$ and $M = G$. Then, taking into account Eq. (20), the disturbing potential Eq. (9) is written $T : \langle\mathcal{V}\rangle_\mu \equiv U$

$$U = \frac{n^2}{16}\,(2 - 3s_I^2 + 3s_I^2\cos 2\phi)\left\{(2C - B - A)\left[1 - 3\,\frac{f}{f+m}\,\mathrm{dn}^2(u|m)\right]\right.$$
$$\left. +3\,(B - A)\left[1 - \frac{f}{f+m}\,\mathrm{dn}^2(u|m)\right]\left[1 - 2\,\frac{(1+f)\,\mathrm{sn}^2(u|m)}{1 + f\,\mathrm{sn}^2(u|m)}\right]\right\} \tag{25}$$

where $\phi = h - \theta$, and m, and therefore u, still remain to be defined.

Rotating frame

We note that ϕ is the argument of the ascending node of the invariant plane with respect to the inertial plane, in a rotating frame with orbital rate $\mathrm{d}\theta/\mathrm{d}t = n$. Therefore, the explicit appearance of the time can be avoided by moving to the rotating frame by introducing the Coriolis term $-n\,\Phi$, where $\Phi \equiv \Lambda = H$ is now the conjugate momenta of ϕ. Then,

$$\mathcal{K} = \mathcal{K}_0 - n\,\Phi + U \tag{26}$$

where we take the Coriolis term to be of first order and the gravity-gradient potential U to be a second order quantity.

PERTURBATION APPROACH

We use the standard Lie transforms procedure[21–23] to find the secular terms of the Hamiltonian. Thus, starting from a Hamiltonian expanded as a power series of a small parameter ε: $\mathcal{K} = \sum_{i\geq 0}(\varepsilon^i/i!)H_{i,0}$, we find the canonical transformation that, up to a truncation order, reduces this Hamiltonian to its secular terms $\mathcal{S} = \sum_{i\geq 0}(\varepsilon^i/i!)H_{0,i}$. The transformation is computed from a generating function $W = \sum_{i\geq 0}(\varepsilon^i/i!)W_{i+1}$. This is done in a stepwise procedure that is usually known as "filling the Lie triangle" in which each order of the new Hamiltonian $H_{0,i}$ is selected at will, commonly from an averaging of previous terms, while the corresponding order of the generating function W_i is solved from a partial differential equation. The result is a new Hamiltonian and a generating function from which the transformation equations are computed in a new application of the Lie triangle.[22]

In our case, we assume that the gravity-gradient torque is of higher order than the Coriolis term, which in turn is of higher order than torque-free rotation. This ordering of the Hamiltonian allows us to split the averaging procedure into two parts.

Average over ℓ

First, we look for a canonical transformation $(\ell, g, \phi, L, G, \Phi) \xrightarrow{T_\ell} (\ell', g', \phi', L', G', \Phi')$ that removes the variable ℓ from the Hamiltonian. We set

$$H_{0,0} = \mathcal{K}_0, \qquad H_{1,0} = -n\,\Phi, \qquad H_{2,0} = 2\,U,$$

where all the functions are assumed to be expressed in prime variables. For the sake of brevity, we drop the prime notation in what follows when there is no risk of confusion.

The first step in the computation of the Lie triangle gives

$$\frac{G}{2A}\,\frac{C-A}{C}\,\frac{f}{(f+m)^2}\,\frac{\partial m}{\partial \beta}\,\frac{\partial W_1}{\partial \ell} = H_{1,0} - H_{0,1}$$

Because $H_{1,0}$ does not depend on ℓ or g, we choose $H_{0,1} = H_{1,0}$, and the first term in the generating function is $W_1 = 0$. Because the vanishing of W_1, the next step gives

$$\frac{G}{2A}\,\frac{C-A}{C}\,\frac{f}{(f+m)^2}\,\frac{\partial m}{\partial \beta}\,\frac{\partial W_2}{\partial \ell} = H_{2,0} - H_{0,2} \tag{27}$$

We choose $H_{0,2} = \langle H_{2,0}\rangle_\ell$ where recalling that the Jacobian elliptic functions are $4\mathrm{K}(m)$-periodic,

$$H_{0,2} = \frac{1}{T}\int_0^T H_{2,0}\,\mathrm{d}\ell = \frac{1}{4\mathrm{K}(m)}\int_0^{4\mathrm{K}(m)} H_{2,0}\,\mathrm{d}u. \tag{28}$$

Remarkably, the averaged Hamiltonian is computed without need of defining yet the transformation derived from the Hamilton-Jacobi equation. We get

$$H_{0,2} = \frac{n^2}{4}\,\kappa\,(2 - 3s_I^2 + 3s_I^2\cos 2\phi), \tag{29}$$

where[‡]

$$\kappa = (B-A)\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{m+f}\left[1 + \frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\}, \tag{30}$$

where $\mathrm{E}(m)$ is the complete elliptic integral of the second kind. In view of $m = m(L, G)$ and $f = f(A, B, C)$, we note that κ only depends on the new momenta L and G and on the inertia parameters.

Then, the term W_2 of the generating function can be solved from Eq. (27), still without need of choosing m. We get

$$W_2 = -\frac{3}{2}\,\frac{n^2}{G}\,(C-B)\,A\sqrt{f\,\frac{1+f}{f+m}}\;\mathrm{Z}(\psi|m)\left(1 - 3s_I^2\sin^2\phi\right) \tag{31}$$

‡The function κ first appeared in Ref. 2 (Eq. (12), p. 147). Note that in Chernous'ko's notation $C \le B \le A$, and $m = k^2$.

where

$$\mathrm{Z}(\psi|m) = \mathrm{E}(\psi|m) - \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\,\mathrm{F}(\psi|m) = \mathrm{E}(\mathrm{am}(u|m)\,|\,m) - \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\,u \tag{32}$$

is the Jacobi zeta function, which is periodic and plays an analogous role to the "equation of the center" of the orbital motion. $\mathrm{E}(\psi|m)$ is the incomplete elliptic integral of the second kind.

Once the generating function has been computed, one may be tempted to improve the secular terms by computing higher orders of the averaged Hamiltonian. The construction of the Lie-triangle only involves the trivial computation of Poisson brackets. Nevertheless, we immediately find non-trivial integrals related to the "equation of the center", Eq. (32).

The choice of the transformation from the Hamilton-Jacobi family

We remark that in spite of the Jacobi zeta function is periodic, its derivative with respect to the elliptic parameter may be periodic or not: it depends on the specific choice of $X(m)$, cf. Eq. (19), which in turn defines m as the solution of a differential equation.

For the sake of illustrating the whole procedure, we chose here a specific transformation. First of all, we note that we can proceed in two different ways. On one side, we can impose some condition either to Eq. (11) or to Eq. (12) and solve m from a differential equation. Alternatively, we can enforce two simultaneous conditions to Eq. (11) and (12), and then compute $m = m(\beta)$ by simply eliminating $\partial m/\partial\beta$ from an algebraic system.

For instance, the simultaneous conditions

$$X \equiv \frac{1}{2}\,\frac{\sqrt{f\,(1+f)}}{(f+m)^{3/2}}\,\frac{\partial m}{\partial\beta} = -1, \qquad Y \equiv \frac{1}{f+m}\left(m - \frac{f}{f+m}\,\frac{\beta}{2}\,\frac{\partial m}{\partial\beta}\right) = 1,$$

imposed to Eq. (11) and (12), respectively, allow for the trivial elimination of $\partial m/\partial\beta$, leading to

$$m = f\left[(1+f)\,\frac{G^2}{L^2} - 1\right], \tag{33}$$

and, consequently,

$$\mathcal{K}_0 = \frac{G^2}{2A} - \left(\frac{1}{B} - \frac{1}{C}\right)\frac{L^2}{2}. \tag{34}$$

In addition, both the direct and inverse transformation are given explicitly, cf. Ref. 8. A drawback of this transformation might be that neither ℓ nor g are angles.

On the other hand, the simultaneous conditions

$$X \equiv \frac{1}{2}\,\frac{\sqrt{f\,(1+f)}}{(f+m)^{3/2}}\,\frac{\partial m}{\partial\beta} = -\frac{\pi}{2\mathrm{K}(m)}, \tag{35}$$

$$Y \equiv \frac{1}{f+m}\left(m - \frac{f}{f+m}\,\frac{\beta}{2}\,\frac{\partial m}{\partial\beta}\right) = \frac{\Pi(-f,m)}{\mathrm{K}(m)}, \tag{36}$$

imposed to Eq. (11) and (12), respectively, render both ℓ and g angles. But the resulting transformation has the inconvenience of giving the elliptic parameter as an implicit function of the new momenta.[17] Thus, the trivial elimination of $\partial m/\partial\beta$ from Eqs. (35) and (36) gives

$$\frac{L}{G}=\frac{2}{\pi}\sqrt{\frac{1+f}{f}\,(f+m)}\left[\Pi(-f,m)-\frac{m}{f+m}\,\mathrm{K}(m)\right].$$

Hence, the Hamiltonian in the new variables must remain in the standard form of Eq. (18).

In what follows we base on the first transformation defined by Eqs. (33) and (34) because of the explicit character of the transformation and its analogy with Hitzl and Breakwell's proposal.[4] Therefore, $u=\ell=-F(\psi|m)$.

Transformation equations of the first averaging

Because $m\equiv m(G,L)$, it worths to recall that

$$\begin{aligned}\frac{\partial\mathrm{E}(m)}{\partial(L,G)} &= \frac{\partial\mathrm{E}(m)}{\partial m}\,\frac{\partial m}{\partial(L,G)}=\frac{1}{2m}\left[\mathrm{E}(m)-\mathrm{K}(m)\right]\,\frac{\partial m}{\partial(L,G)},\\ \frac{\partial\mathrm{K}(m)}{\partial(L,G)} &= \frac{\partial\mathrm{K}(m)}{\partial m}\,\frac{\partial m}{\partial(L,G)}=\frac{1}{2m}\left[\frac{1}{1-m}\,\mathrm{E}(m)-\mathrm{K}(m)\right]\,\frac{\partial m}{\partial(L,G)},\end{aligned}$$

where, cf. Eq. (33),

$$G\,\frac{\partial m}{\partial G}=-L\,\frac{\partial m}{\partial L}=2\,(m+f). \tag{37}$$

In this case, the transformation equations are

$$\xi=\xi'+\frac{1}{2}\,\{\xi';W_2\},\qquad \xi\in(\ell,g,\phi,L,G,\Phi)$$

where $\{a;b\}$ stands for the Poisson bracket of the functions a and b.

Then, calling $\Delta\xi=\xi-\xi'$, we find

$$\Delta\ell = \frac{3n^2}{4G^2}\,A\,(C-B)\left[2(f+m)\,\frac{\partial\mathrm{Z}(\psi|m)}{\partial m}-\mathrm{Z}(\psi|m)\right]\left(1-3s_I^2\sin^2\phi\right) \tag{38}$$

$$\Delta g = -\frac{L}{G}\left[\Delta\ell-\frac{3n^2}{4G^2}\,A\,(C-B)\,\mathrm{Z}(\psi|m)\left(1-3s_I^2\sin^2\phi+6c_I^2\sin^2\phi\right)\right] \tag{39}$$

$$\Delta\phi = -\frac{3n^2}{4G^2}\,A\,(C-B)\,\frac{L}{G}\,\mathrm{Z}(\psi|m)\,6c_I\sin^2\phi \tag{40}$$

$$\Delta L = \frac{3n^2}{4G^2}\,A\,(C-B)\,L\left[\frac{\mathrm{E}(m)}{\mathrm{K}(m)}-\mathrm{dn}(\ell|m)^2\right]\left(1-3s_I^2\sin^2\phi\right) \tag{41}$$

$$\Delta G = 0 \tag{42}$$

$$\Delta\Phi = -\frac{3n^2}{4G^2}\,A\,(C-B)\,L\,\mathrm{Z}(\psi|m)\,3s_I{}^2\sin 2\phi \tag{43}$$

where we remind that the right member of the equations should be assumed in the prime variables.

The derivative of the equation of the center with respect to the elliptic modulus required by $\Delta\ell$ in Eqs. (38)–(39) is

$$\frac{\partial}{\partial m}\,\mathrm{Z}(\mathrm{am}(\ell|m)\,|\,m) = \frac{\partial}{\partial m}\,\mathrm{E}(\mathrm{am}(\ell|m)\,|\,m) - \ell\,\frac{\partial}{\partial m}\frac{\mathrm{E}(m)}{\mathrm{K}(m)}$$

where

$$\begin{aligned}
\frac{\partial}{\partial m}\frac{\mathrm{E}(m)}{\mathrm{K}(m)} &= \frac{1}{2m}\left[2\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)} - \frac{\mathrm{E}^2(m)}{(1-m)\,\mathrm{K}^2(m)} - 1\right],\\
\frac{\partial}{\partial m}\,\mathrm{E}(\mathrm{am}(\ell|m)\,|\,m) &= \frac{1}{2(1-m)}\Big[\mathrm{dn}(\ell|m)\,\mathrm{cn}(\ell|m)\,\mathrm{sn}(\ell|m)\\
&\qquad - \mathrm{cn}^2(\ell|m)\,\mathrm{E}(u|m) - (1-m)\,\mathrm{sn}^2(\ell|m)\,\ell\Big],
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial m}\,\mathrm{Z}(\psi|m) &= \frac{\mathrm{cn}(\ell|m)}{2\,(1-m)}\Big[\mathrm{dn}(\ell|m)\,\mathrm{sn}(\ell|m) - \mathrm{cn}(\ell|m)\,\mathrm{Z}(\psi|m)\Big]\\
&\quad -\frac{1}{2m}\left[1 - \frac{1}{1-m}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\left[\frac{\mathrm{E}(m)}{\mathrm{K}(m)} - \mathrm{dn}^2(\ell|m)\right]\ell
\end{aligned} \tag{44}$$

that is not periodic, as illustrated in the right plot of Fig. 1. This means that the transformation equations for the variables ℓ and g are affected of mixed terms.

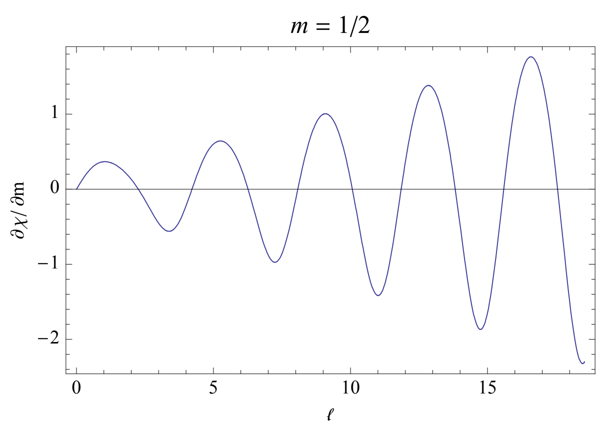

Figure 1. Derivative of the center equation $\chi = Z\big(\mathrm{am}(\ell\,|\,m)|m\big)$ with respect to the modulus in the case $m = 1/2$.

Nevertheless, some cases can be found in which this derivative is also a periodic function, thus avoiding the appearance of mixed terms in the transformation equations. That is the case of action-angle variables,[11] for which

$$\frac{\partial}{\partial m}\,\mathrm{Z}(\psi|m) = \frac{\mathrm{cn}(u|m)}{2\,(1-m)}\Big[\mathrm{dn}(u|m)\,\mathrm{sn}(u|m) - \mathrm{cn}(u|m)\,\mathrm{Z}(\psi|m)\Big] \tag{45}$$

where $u = 2\mathrm{K}(m)\,\ell/\pi$. The closed form integration of the tumbling satellite problem in action-angle variables is under progress and will be presented elsewhere.

Average over ϕ

A new Lie transform $(\ell', g', \phi', L', G', \Phi') \longrightarrow (\ell'', g'', \phi'', L'', G'', \Phi'')$ such that it removes the angle ϕ is then computed. We start by setting $K_{0,0} = H_{0,0}$, $K_{1,0} = H_{0,1}$, and $K_{2,0} = H_{0,2}$, all of them evaluated in the double prime variables although we drop the primes from the notation for brevity.

Because the only angle that appears in the Hamiltonian is ϕ'', we may assume that the new generating function $V = \sum_{i\geq 0}(\varepsilon^i/i!)\, V_{i+1}$ only depends on this angle. Then, the first step in the Lie triangle gives $0 = K_{1,0} - K_{0,1}$ and we trivially choose $K_{0,1} = K_{1,0}$ while V_1 remains unknown at this step. The second step gives

$$K_{0,2} = 2\,n\,\frac{\partial V_1}{\partial \phi} + K_{2,0}. \tag{46}$$

We choose $K_{0,2} = \frac{1}{2\pi}\int_0^{2\pi} K_{2,0}\,\mathrm{d}\phi$, which produces

$$K_{0,2} = \frac{n^2}{4}\left(3\frac{\Phi^2}{G^2} - 1\right)\kappa. \tag{47}$$

Then, from Eq. (46) $V_1 = \frac{1}{2\,n}\int (K_{0,2} - K_{2,0})\,\mathrm{d}\phi$, resulting in

$$V_1 = -\frac{3n}{16}\left(1 - \frac{\Phi^2}{G^2}\right)\kappa\sin 2\phi. \tag{48}$$

Transformation equations of the second averaging

Now, the transformation equations are

$$\xi' = \xi'' + \{\xi''; V_1\}, \qquad \xi \in (\ell, g, \phi, L, G, \Phi).$$

Then, calling $\delta\xi = \xi' - \xi''$, we find

$$\delta\ell = \frac{n}{2L}\,\frac{9}{4}\,A\,\frac{C-B}{C}\left(1 - \frac{\Phi^2}{G^2}\right)\frac{L^2}{G^2} \tag{49}$$
$$\times\left\{1 + \frac{C-B}{B}\,\frac{f+m}{2m}\left[1 - \frac{2f}{f+m}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)} + \frac{1}{1-m}\,\frac{\mathrm{E}^2(m)}{\mathrm{K}^2(m)}\right]\right\}\sin 2\phi$$

$$\delta\phi = \frac{3n}{8\Phi}\,\frac{\Phi^2}{G^2}\,\kappa\sin 2\phi \tag{50}$$

$$\delta g = -\frac{\Phi}{G}\,\delta\phi - \frac{L}{G}\,\delta\ell \tag{51}$$

$$\delta L = 0 \tag{52}$$

$$\delta G = 0 \tag{53}$$

$$\delta\Phi = \frac{3n}{8}\left(1 - \frac{\Phi^2}{G^2}\right)\kappa\cos 2\phi \tag{54}$$

where the right member of the equations must be expressed in the double-prime variables.

Secular terms

After the double averaging we find the secular Hamiltonian $\mathcal{S} = K_{0,0} + K_{0,1} + \frac{1}{2}K_{0,2}$ given by

$$\begin{aligned}\mathcal{S} \;=\; & \frac{G^2}{2A} - \left(\frac{1}{B} - \frac{1}{C}\right)\frac{L^2}{2} - n\,\Phi \\ & -\frac{n^2}{8}\left(1 - 3\frac{\Phi^2}{G^2}\right)(B-A)\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{f+m}\left[1 + \frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\}\end{aligned} \tag{55}$$

where L, G, and therefore m, and Φ are dynamic integrals. The secular frequencies of the motion are obtained from Hamilton equations,

$$\begin{aligned}\frac{\mathrm{d}\ell}{\mathrm{d}t} = & -\left(\frac{1}{B} - \frac{1}{C}\right)L + L\,\frac{3n^2}{4G^2}\,A\,\frac{C-B}{C}\left(1 - 3\frac{\Phi^2}{G^2}\right) \\ & \times\left\{1 - \frac{C-B}{B}\,\frac{f}{m}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)} + \frac{C-B}{B}\,\frac{f+m}{2m}\left[1 + \frac{1}{1-m}\,\frac{\mathrm{E}^2(m)}{\mathrm{K}^2(m)}\right]\right\} = n_\ell\end{aligned} \tag{56}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -n + \Phi\,\frac{3n^2}{4G^2}(B-A)\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{f+m}\left[1 + \frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\} = n_\phi \tag{57}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{1}{A}\,G - \frac{\Phi}{G}\,(n_\phi + n) - \frac{L}{G}\left[n_\ell + \left(\frac{1}{B} - \frac{1}{C}\right)L\right] = n_g \tag{58}$$

and $\mathrm{d}h/\mathrm{d}t = n_h = n + n_\phi$.

SAMPLE APPLICATION

To illustrate the application of the theory and show the improvements obtained when adding the periodic terms to the secular frequencies propagation, we provide a sample application. We base on the orbit and inertia parameters of a PEGASUS-A satellite, which we take from Ref. 5 except for we consider a Keplerian circular orbit, in agreement with the assumptions of our theory. Thus,

$$A = 1.03068 \times 10^5\,\mathrm{kg\,m^2}, \quad B = 3.33455 \times 10^5\,\mathrm{kg\,m^2}, \quad C = 3.94992 \times 10^5\,\mathrm{kg\,m^2}$$

and

$$M = 5.842 \times 10^5\,\mathrm{kg\,m^2/min}, \quad n = 3.71^\circ/\mathrm{min}$$

The phase space of the Euler-Poinsot problem for these momenta of inertia is illustrated in Fig. 2. Since the perturbation theory has averaged ν, it is only valid in those regions in which this angle circulates. Therefore, we choose the initial conditions in the region where the trajectories $J = J(\nu)$ exist in all the range $0 \le \nu \le \pi$. Specifically, we choose the initial conditions $\mu = 2\,\mathrm{rad}$, $\nu = 1\,\mathrm{rad}$, $\lambda = -0.1\,\mathrm{rad}$, $J = 10^\circ$, $i = 70^\circ$. The values of ν and J are represented by the big dot in the left-bottom part of Fig. 2.

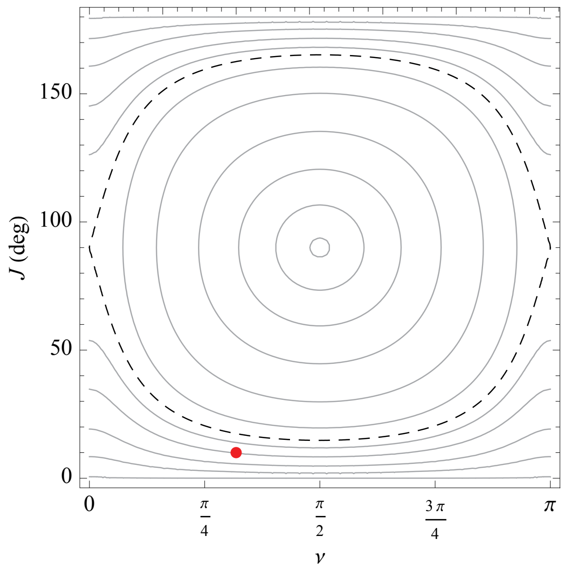

Figure 2. Phase space of the Euler-Poinsot problem of a PEGASUS-A satellite.

For the integration we use internal units such that $M = C = 1$. Then, using Eq. (11)–(17) we transform the initial conditions in Andoyer variables to

$$\begin{aligned} \ell_0 &= -0.1626833314, \\ g_0 &= 2.0665318080, \\ h_0 &= -0.1, \\ L_0 &= 3.8744459575, \\ G_0 &= 1, \\ H_0 &= 0.3420201433. \end{aligned} \tag{59}$$

and $\Phi_0 = H_0$. We assume that $\theta = 0$ at $t = 0$ and, therefore, $\phi_0 = h_0$. A net transformation of the initial conditions to the single-averaged phase using Eqs. (38)–(43) space gives

$$\begin{aligned} \ell_0' &= -0.1628298853, \\ g_0' &= 2.0670936407, \\ \phi_0' &= -0.0999998751, \\ L_0' &= 3.8744812340, \\ G_0' &= G_0, \\ \Phi_0' &= 0.3420169296. \end{aligned} \tag{60}$$

Finally, using Eqs. (49)–(54) we obtain the initial conditions

$$\begin{aligned} \ell_0'' &= -0.1592197766, \\ g_0'' &= 2.0534303122, \\ \phi_0'' &= -0.1009172983, \\ L_0'' &= L_0', \\ G_0'' &= G_0, \\ \Phi_0'' &= 0.3531301948, \end{aligned} \tag{61}$$

and secular frequencies

$$\begin{aligned} n_{\ell''} &= -0.7146350295, \\ n_{g''} &= 3.8310289891, \\ n_{\phi''} &= -0.0441809428, \end{aligned} \tag{62}$$

that are computed from Eqs. (56)–(57).

The initial conditions in the double-averaged phase space, Eq. (61), are propagated for several orbital periods of the PEGASUS-A-type satellite using the secular frequencies, Eq. (62). Results are compared with the propagation of the Hamilton equations of the non-averaged Hamiltonian, Eq. (26), for corresponding initial conditions in the non-averaged phase space, Eq. (60).

The attitude evolution in the non-averaged phase space is presented in Fig. 3, where to better appreciate details introduced by the gravity-gradient torque, we subtract to each variable the constant rate of the torque-free motion, represented by the tilde variables. From Hamilton equations derived from $\mathcal{K}_0 - n\,\Phi$, with $\mathcal{K}_0$ given in Eq. (18), we find: $\tilde{L} = L_0$, $\tilde{G} = G_0$, $\tilde{\Phi} = \Phi_0$, and

$$\tilde{\ell} = \ell_0 - \left(\frac{1}{B} - \frac{1}{C}\right) L_0\, t, \tag{63}$$

$$\tilde{g} = g_0 + \frac{1}{A} G_0\, t, \tag{64}$$

$$\tilde{\phi} = \phi_0 - n\, t. \tag{65}$$

Thus, we find that ℓ advances over the torque-free motion with a constant rate of about 0.053 units times orbital period, and is affected of short and long-period effects. On its side, g slows down with respect the torque-free motion at a much higher rate of about -0.183 radians (or ~ 10.5 deg) times orbital period, plus periodic effects. The rate of departure of ϕ from the torque-free motion is about -3.3 deg times orbital period and is only affected of periodic terms related to the orbital motion. Finally, L differs from the constant torque-free motion value only in low-amplitude periodic terms, while Φ advances over the unperturbed value at a small rate of 0.002 units times orbital period and similarly to its conjugate variable ϕ, is only affected of long-period effects related to the orbital motion.

The errors of the propagation of the secular terms are presented in Fig. 4, where we note that the propagation of the secular terms is affected of short-period oscillations related to the rotational motion and long-period oscillations with half the orbital period. These periodic terms mask a secular trend related to the truncation of the Lie-transforms theory.

On the contrary, if we recover the periodic terms using first Eqs. (49)–(54) to compute the prime elements $x' = x'' + \delta x''$, $x \in (\ell, g, \phi, L, G, \Phi)$, and then Eqs. (38)–(43), we obtain a set of elements $x^* = x' + \Delta x'$ that are much closer to the original solution. The errors of this set of approximate elements are shown in Fig. 5. Thus, when comparing Figs. 5 and 4 we note that, in spite of both short and long-period errors remain in Fig. 5 they are now much smaller, clearly disclosing the secular error due to the truncation of the perturbation theory.

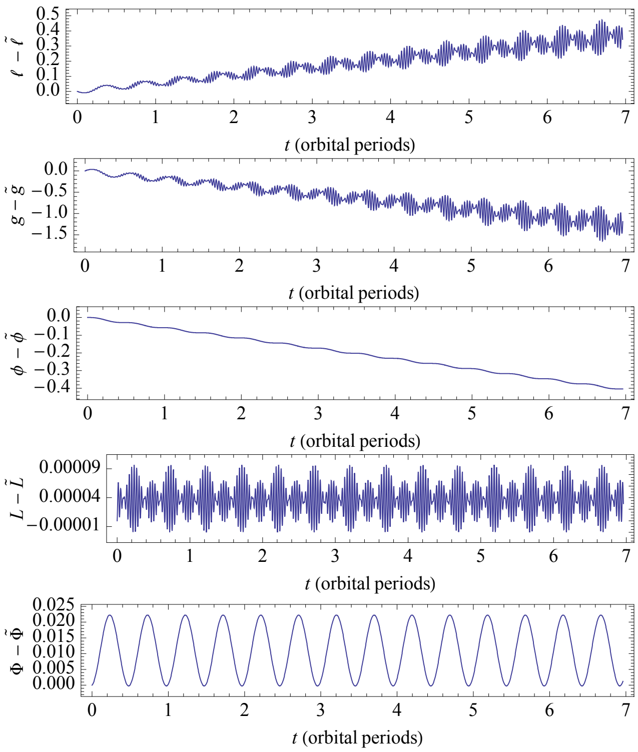

Figure 3. Differences between the torque-free and perturbed motions (internal units).

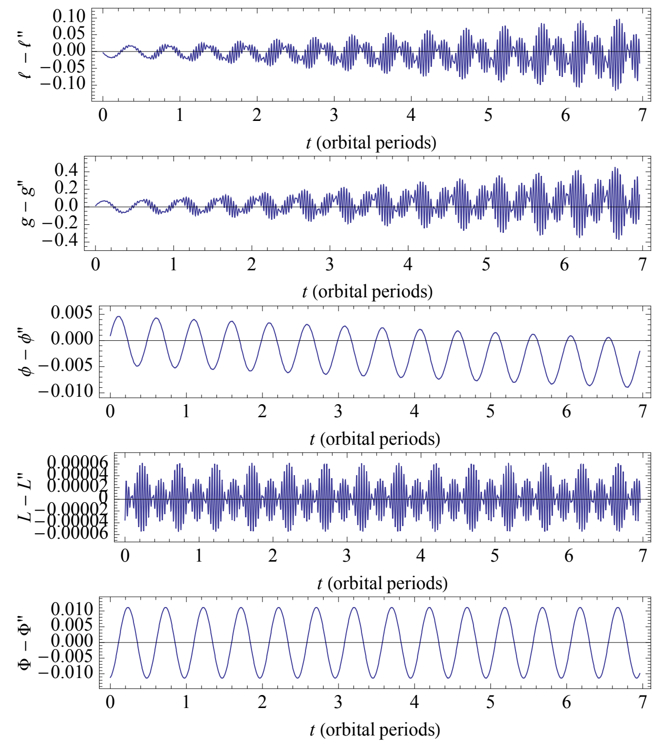

Figure 4. Errors of the secular terms propagation (internal units).

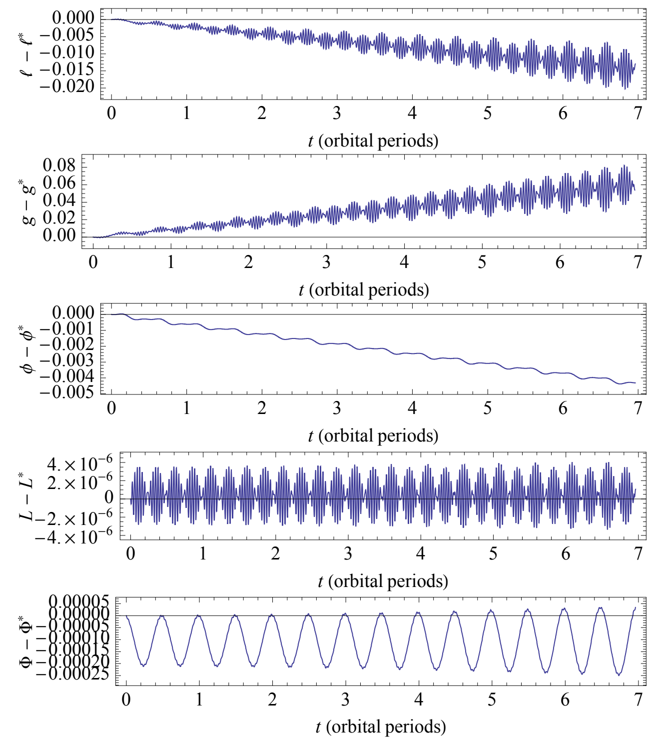

Figure 5. Same as Fig. 4 with $x^* = x' + \Delta x'$, and $x' = x'' + \delta x''$, $x \in (\ell, g, \phi, L, G, \Phi)$.

CONCLUSIONS

The tumbling satellite problem for the attitude propagation of triaxial satellites has been revisited and improved with the computation of the short periodic terms. The perturbation solution relies in a set of variables that provide complete reduction of the Euler-Poinsot problem when expressed in Andoyer variables. This reduction is done formally without need of choosing in advance any specific form of the reduced Hamiltonian. This fact makes that the secular terms of the tumbling satellite problem, as well as the generating function, are computed in a general form that either does not need the previous selection of the canonical variables used in the procedure. Once a particular set of variables is chosen, one can easily achieve the computation of the long and short periodic corrections that are related to the orbital motion and to the rotation of the body frame about the angular momentum vector, respectively. These corrections are provided as a compact set of formulas which are computed in closed form of the elliptic parameter.

Proceeding this way, it is shown the fundamental role that the Jacobi zeta function plays in the perturbation solution. Since the Jacobi zeta function appears as soon as in the early stages of the perturbation approach, one may anticipate non-trivial difficulties when trying to extend the closed form integration of the tumbling satellite problem to higher orders. These problems would be similar in nature to those arising from the integration of the equation of the center in the satellite problem, so one would expect that succeeding with the closed form integration to higher orders, if possible, will require the use of special functions of the polylogarithmic type.

Following previous approaches to the tumbling satellite problem, we have chosen a set of canonical variables that reduce the Euler-Poinsot Hamiltonian to a quadratic form. The drawback of these variables is that they introduce mixed terms in the transformation equations of the perturbed problem. In spite of this undesired effect may be not important in the time scales used in the examples, a remedy is found in the use of action-angle variables. On the other side, the action-angle variables have the drawback of requiring the inversion of implicit equations in the solution procedure.

Despite we restricted the gravity-gradient torque to the simpler case of a satellite in circular orbit, the mathematical procedures employed are not limited in application to this simple model, and the theory can be extended to more real models that include, for instance, the case of elliptic orbits and the effect of the Earth's oblateness in the motion of the orbital node and perigee.

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