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A NEW APPROACH ON THE LONG TERM DYNAMICS OF NEO'S UNDER YARKOVSKY EFFECT

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A classical approach to the many-body problem is that of using special perturbation methods. Nowadays and due to the availability of high-speed computers is an essential tool in Space Dynamics which exhibits a great advantage: it is applicable to any orbit involving any number of bodies and all sorts of astrodynamical problems, especially when these problems fall into regions in which general perturbation theories are absent. One such case is, for example, that Near Earth Objects (NEO's) dynamics. In this field, the Group of Tether Dynamics of UPM (GDT) has developed a new regularization scheme —called DROMO— which is characterized by only 8 ODE. This new regularization scheme allows a new approach to the dynamics of NEO's in the long term, specially appropriated to consider the influence of the anisotropic thermal emission (Yarkovsky and YORP effects) on the dynamics. A new project, called NEO-DROMO, has been started in GDT that aims to provide a reliable tool for the long term dynamics of NEO's.

INTRODUCTION

Some Near-Earth Objects (NEO's) of the Solar System mean a real threat for the life on Earth. The geological and biological history of our planet is punctuated by evidence of repeated, devastating cosmic impacts. References [1, 2] deal with the asteroid threats in a detailed way; also, the volume 2 of the *Journal of Cosmology* is entirely devoted to these kind of threats. In Ref. [1] we can read:

International NEO decision-making should take the following factors into consideration:

- *Damage caused by asteroids and other Near Earth Objects might affect the entire international community and/or major parts of the world. A truly global response is required.*
- *Capabilities (unevenly spread among the international community) are available to humankind to undertake responsive action against NEO threats, especially if the appropriate decisions are made sufficiently in advance.*
- *The discovery rate of NEOs posing a potential threat will increase significantly within the next 10-15 years.*
- *Because a substantial lead time is usually required to execute an asteroid deflection operation, the international community may have to act before it would be certain an impact would occur.*

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- *Efforts to deflect a NEO could cause a temporary shift in the impact site from one populated region of the planet to another.*
- *Delays in decisions to undertake responsive actions will limit the relevant options. Such delays will increase the risk that the remaining options may cause undesirable political consequences or even physical impact damage.*

As a consequence, asteroid deflection is becoming a key topic in astrodynamics. Although no asteroid has been deflected so far, altering the trajectory of a small-sized asteroid to avoid a catastrophic impact with the Earth has been shown to be, in principle, technically feasible [3], and different techniques, ranging from nuclear detonation to kinetic impact and low-thrust methods, have been proposed [3, 4, 5]. Each one of these methods shows advantages and drawbacks that, in general, depend on the mass and orbital characteristics of the particular asteroid to be deflected as well as its physical property (porosity, composition, surface reflectivity, etc.) and rotation state.

Recently a new concept, the **Ion Beam Shepherd (IBS)**, has been introduced by our group[1]. This concept allows the introduction of a controlled force on a spacecraft, or a celestial body, by means of a highly collimated high-velocity ion beam which is produced by an ion thruster onboard a shepherd spacecraft. The ion beam is pointed against a target to modify its orbit and/or attitude with no need for docking. It can be used to deflect a threatening asteroid (see [6]) or to remove space debris (see [7]). In the context of the *Ariadna Call for Ideas: Active Removal of Space Debris*, the IBS concept has been partially developed in Ref. [8], a project carried out in collaboration with the EP2 team[2] of the UPM.

Regardless of the chosen defection method, accurate orbit propagation and determination is of paramount importance in any deflection mission. However, when using low-trust deflexion techniques the accurate knowledge of the asteroid dynamics turns out to be a key point of the mission, since a substantial lead time should be provided in order to execute the deflection operation with the required reliability.

The dynamics of a NEO, specially in the long term, is always a n-body problem with additional perturbations. A classical approach to the many-body problem is that of using **special perturbation methods**. Nowadays, and due to the availability of high-speed computers, special perturbation methods are an essential tool in space dynamics which exhibit a great advantage: they are applicable to any orbit involving any number of bodies and all sorts of astrodynamical problems, especially when these problems fall into regions in which general perturbation theories are absent. This is the case of NEO's dynamics.

Two-body regularization is an efficient tool to integrate perturbed two-body problems numerically. This is true not only in Keplerian astrodynamics but also in n-body simulations. Originally regularization was developed to avoid the numerical difficulty in integrating nearly parabolic orbits such as those of comets. However, its effectiveness was confirmed even for nearly circular orbits due to its better numerical stability than unregularized Keplerian motion [9]. A numerical comparison of different *regularized* schemes together with the unregularized formulation in the light of their computational cost and performance can be found in [10].

Our group has developed a new regularization scheme —called DROMO— which is characterized by only 8 ODE. This special perturbation method was presented for the first time in the 2005 winter meeting of the AAS [11], but the basic theory of DROMO can be found in [12] that was published in 2007 almost simultaneously with the Fukushima report (DROMO is not evaluated in [10]).

This novel method is especially appropriated to carry out the propagation of **complex orbits**, like, for example, NEO's orbits. The formulation of DROMO is flexible and it permits, in some cases, to obtain analytical or semi-analytical solutions; an example of this flexibility can be found in [13] where a new asymptotic solution has been obtained for the constant tangential thrust acceleration case. However, the best performances of DROMO are obtained when it is used in the numerical propagation of orbits. Thus, DROMO turns out to be one of the most accurate propagators when compared with similar formulations. Due to the plus of accuracy provided by the DROMO formulation (see section 4 in page 5), this scheme is quite appropriated for the propagation of orbits when a high-fidelity description of the trajectory is mandatory. This tool will be used to study the dynamical behavior of small celestial bodies.

Aside from the gravitational perturbations, NEO's dynamics is affected by non-conservative perturbations of thermal origin. The effects of thermal radiation forces and torques on small bodies of our Solar System have

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been extensively studied during the last decades. An excellent description of these phenomena can be found in two Ph.D. thesis defended in Charles University (see [14, 15]) focusing on its influence on the dynamics of asteroids and small bodies. The main effects of these perturbations of thermal origin are called the **Yarkovsky effect** and the **YORP effect** (Yarkovsky - O'Keefe - Radzievskii - Paddack).

YARKOVSKY AND YORP EFFECTS

The Yarkovsky effect is a tiny nongravitational force due to radiative recoil of the anisotropic thermal emission. When the temperature of the surface of the body is not uniform the radiation emitted by the body varies from one point to another. This causes a force —responsible for the Yarkovsky effect— and a torque —responsible for the YORP effect— acting on the body.

In general, the Yarkovsky effect induces a secular deviation on the semimajor axis of a heliocentric orbit of asteroids of appropriate size. In addition, the forces associated to this thermal phenomenon also modifies the rotational dynamics of the asteroid, due to the non-vanishing torque on its center of mass; this additional effect —commonly called the YORP effect— is responsible for the variation of the attitude dynamics of the body due to thermal effects.

These effects are important in the long-term because, unlike gravitational perturbations, they can permanently increase or decrease orbital and/or rotational energy. As a consequence of the secular evolution of the semimajor axis the body may migrate from one heliocentric zone to another. Similarly, rotation rate and obliquity of the spin axis could be permanently changed such that a normal rotator may be moved to the category of fast or slow, or even tumbling, rotators.

The intensity of the coupling between orbital and attitude motion depends on many factors; the size is one of the most important together with the spin of the asteroid. Thus, the above mentioned coupling cannot be neglected in some asteroids and a joint description is mandatory in order to predict its orbit with accuracy.

To properly calculate the force associated with this particular effect the temperature distribution on the surface of the asteroid should be determined; but this temperature depends on the asteroid orbit, its size and shape, spin axis orientation and period, mass, density of surface layers, albedo, thermal conductivity, capacity and IR emissivity of the material (see [16]). The uncertainty of many of these parameters invites to develop simplified methods to calculate the influence of the thermal effects on long term dynamics of asteroids as, for example, the excellent paper [17].

The elemental force $\mathrm{d}\,\vec{\boldsymbol{f}}$ due to thermal radiation on a surface element of area $\mathrm{d}S$ is normal to the surface and takes the value

$$\mathrm{d}\,\vec{\boldsymbol{f}} = -\frac{2\varepsilon\sigma T^4}{3c}\,\vec{\boldsymbol{n}}\mathrm{d}s$$

where ε is the emissivity of the surface, $\sigma = 5.67 \times 10^{-8}$ W/(m^{2}K^4) the Stefan-Boltzman constant, $c = 2.998 \times 10^8$ m/s the speed of light in vacuum, $\vec{\boldsymbol{n}}$ is the unit vector normal to the surface and T the temperature of the surface, which has to be determined in order to integrate the force over the entire surface.

The resultant of these forces and the torque at the center of mass of the body are:

$$\vec{\boldsymbol{f}} = -\frac{2\sigma}{3c}\int_\Sigma \varepsilon T^4\,\vec{\boldsymbol{n}}\mathrm{d}s, \qquad \vec{\boldsymbol{T}} = -\frac{2\sigma}{3c}\int_\Sigma \varepsilon T^4\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{n}}\mathrm{d}s$$

where both integrals should be extended to the external surface Σ of the body. Obviously, a geometric model for the surface should be available. Some preliminary but very interesting results have been obtained assuming simple geometries as in [18].

To obtain the surface's temperature distribution, $T(\vec{\boldsymbol{x}}, t)$, the heat diffusion equation is applied. Assuming that heat conduction is one-dimensional (see [19, 20]), that is, assuming that the main temperature gradients are perpendicular to the body surface, the equation is

$$\rho\, c_t \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})$$

where ρ is the density of the body, c_t its specific thermal capacity and k its thermal conductivity. In this equation z is the depth accounted from the body surface. To solve this partial differential equation, boundary and initial conditions must be given.

The first boundary condition comes from the energy balance equation on the body surface. Thus,

$$\varepsilon\sigma T^4(z=0,t) = k\frac{\partial T}{\partial z}(z=0,t) + \alpha\, E(t)$$

In this equation α is the solar absorptance of the body surface and $E(t)$ the solar irradiation on the surface element. It can be calculated from the solar intensity at the distance d (in AU) between the body and the Sun and θ is the angle between the solar rays and the vector normal to the surface. Thus,

$$E(t) = G_s \left(\frac{d_{SE}}{d}\right)^2 \cos\theta$$

where $G_s = 1366.1$ W/m^2 is the solar constant at the distance between the Earth and the Sun: $d_{SE} = 1$ AU.

The second boundary condition is related to the fact that after certain depth, the effect of the Sun on the body temperature is not perceived. Then

$$\lim_{z\to\infty} \frac{\partial T}{\partial z}(z,t) = 0$$

Regarding, the initial conditions, a periodicity restriction is applied, so that

$$T(z,t) = T(z,t+P)$$

where P is the rotational period of the celestial body.

The solution of the differential equation with the boundary and initial conditions described above is quite complex, even for spherical bodies, mainly due to the non-linearity of the boundary conditions. In some works (see [21, 22]) solutions have found by assuming that the temperature distribution of the body fluctuates about an average value $T = T_{av} + T$ and linearizing the radiation term. But, even in this case, the mathematical apparatus necessary to solve the problem is quite complex. More details about these models can be found in [16, 15]. A model similar to the models described in ,[19, 20] will be used in a first step in NEODROMO.

Impact of Yarkovsky effect on NEO dynamics

Non-gravitational perturbations, regardless being many orders of magnitude weaker than gravity [23], hold keys to fully understand the dynamical evolution of asteroids. These forces produce small but meaningful effects on asteroid orbits and rotation rates over long timescales, which suggests that they should be considered as important as collisions and gravitational perturbations for the understanding of asteroid evolution [24].

For meteoroids and small asteroids in the 10 cm - 10 km range, the principal non-gravitational force and torque arise from an anisotropic thermal emission of the absorbed solar radiation [23].

Historically the Yarkovsky perturbation was believed to be negligible assuming that its acceleration, opposed to the gravitational attraction of the Sun, could be modeled by adjusting the gravitational constant of the Sun, and the asteroids were anyway too large objects. However, it can be proved [25] that none of these conditions is typically satisfied for real NEOs, so the effects of direct solar radiation pressure and reflected radiation should also be taken into account, since may lead to observable displacements of the orbit whenever the asteroid is not spherical or its surface albedo is not homogeneous. The Yarkovsky effect may in fact become important in the dynamics of asteroids, since it leads to long-term perturbations accumulating over long time spans.

The most important implication of the Yarkovsky perturbations is the steady and size-dependent semimajor axis drift as a function of the body's spin, orbit and material properties [25, 26, 24]. This effect results in an orbit displacement that accumulates quadratically with time, compared with the linear perturbations due to the eccentricity and inclination [25].

Besides, further Yarkovsky/YORP-driven processes [23] include: secular changes of the rotational period and obliquity, efficient transport towards low-order resonances, interaction with weaker higher-order resonances, or captures in secular and spin-orbit resonances. According with [27] *two major mechanisms are suspected to alter the rotation rates and states of NEOs once they get into planet crossing orbits: close encounters with the planets and YORP*.

As a significative example to the relevance of non-gravitational perturbations, the Yarkovsky/YORP force could push a 10 meter meteoroid's semimajor axis by 0.1-0.2 AU, before being disrupted by a random collision with another body (see Ref. [23]).

NEODROMO PROJECT

Predicting the trajectory of a given asteroid —we are thinking basically in NEO's— involves the knowledge of several standard models of the solar system that includes the gravity of the Sun, Moon, other planets and the three largest asteroids: Ceres, Vesta and Pallas. The ephemeris of all these celestial bodies are needed and *high fidelity prediction requires high fidelity ephemeris*.

Additional factors influence the long-term predicted trajectory in a substantial way: the spin of the asteroid, its mass, the way it reflects and absorbs sun-light, radiates heat, and the gravitational pull of other celestial bodies passing nearby. Most of these factors are associated with small perturbations which act on the asteroid and produce the slow evolution of its classical elements. However, close encounters with celestial bodies introduce a much faster time scale and the dynamical state of the asteroid could change drastically.

When studying the dynamics of NEO's on long time intervals numerical methods of integration of the equations of motion are practically mandatory since the dynamics of these objects is not easily studied by analytical methods because of large eccentricities and close encounters with planets. In addition, the elements of the NEO orbits are known with some uncertainty due to errors associated with the observations used to determine the orbit or the initial conditions used to start the propagation. The presence of these errors, in some cases, invites the use of probabilistic methods to solve the problem (see, for example, the paper [28]).

However, we adopt another point of view. During the last years, the observation techniques and the procedures used to determine orbits have experienced important improvements (see [29, 30, 31]). This is one of the reason why the number of well known NEO's has increased spectacularly. Similarly, the knowledge about the spin rates of asteroids is increasing steadily (see [32, 33]) and it is natural to assume that this bulk of knowledge will increase even more during the next decades. Some recent examples in this sense can be found in [34, 35].

This paper tries to describe the main characteristics of NEODROMO, which is being developed in our group. The idea of NEODROMO project is to offer a propagation tool, specially tailored for the NEO's dynamics, including models of increasing complexity that can be used for the determination of orbits and the prediction of trajectories. The main objective of this project is to obtain numerically an accurate description of the dynamics of a NEO in the general case in which the thermal radiation induce both, the Yarkovsky and the YORP effects. To do that we use the DROMO propagator together with an attitude propagator (see section 5 in page 10) also developed in our group.

ORBIT PROPAGATION. DROMO

DROMO is based in a new special perturbation method whose theory is developed in papers [11, 12] where a set of non-classical elements $(q_1, q_2, q_3, \varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0, \varepsilon_4^0)$ has been introduced. DROMO provides the time evolution of these elements when the perturbations forces are known.

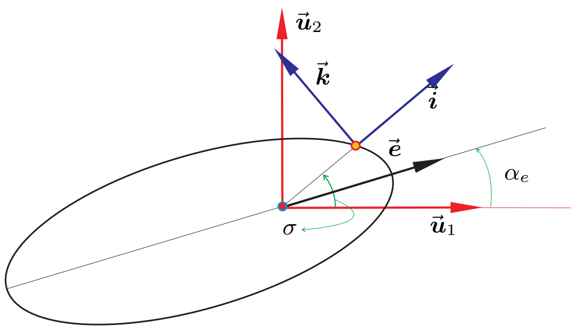

Figure 1: Reference frames

Starting from the original variable, an slight improvement of the performances of DROMO can be obtained by carrying out the following change of variables:

$$\zeta_1 = \frac{q_1}{q_3}, \quad \zeta_2 = \frac{q_2}{q_3}, \quad \zeta_3 = q_3$$

This way the eccentricity vector can be expressed like

$$\vec{e} = \zeta_1 \vec{u}_1 + \zeta_2 \vec{u}_2$$

where the unit vectors $(\vec{u}_1, \vec{u}_2)$, which lie in the orbital plane, are defined by:

$$[\vec{u}_1, \vec{u}_2] = [\vec{i}, \vec{k}]\, Q_0, \qquad Q_0 = \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix}$$

They rotate with angular velocity $+\dot{\sigma}\,\vec{\boldsymbol{j}}$ relative to the orbital frame $(\vec{\boldsymbol{i}},\,\vec{\boldsymbol{k}})$.

Expressed in terms of these new variables the governing equations take the form:

$$\begin{aligned}
\frac{d\zeta_1}{d\sigma} &= \frac{1}{\zeta_3^4\hat{s}^3}\left[+\hat{s}\sin\sigma f_{px} + \{\zeta_1 + (1+\hat{s})\cos\sigma\} f_{pz}\right]\\
\frac{d\zeta_2}{d\sigma} &= \frac{1}{\zeta_3^4\hat{s}^3}\left[-\hat{s}\cos\sigma f_{px} + \{\zeta_2 + (1+\hat{s})\sin\sigma\} f_{pz}\right]\\
\frac{d\zeta_3}{d\sigma} &= -\frac{1}{\zeta_3^3\hat{s}^3} f_{pz}\\
\frac{d\tau}{d\sigma} &= \frac{1}{\zeta_3^3\hat{s}^2}\\
\frac{d\varepsilon_1^0}{d\sigma} &= -\frac{\lambda(\sigma)}{2}\{\sin(\sigma-\sigma_0)\varepsilon_2^0 + \cos(\sigma-\sigma_0)\varepsilon_4^0\}\\
\frac{d\varepsilon_2^0}{d\sigma} &= +\frac{\lambda(\sigma)}{2}\{\sin(\sigma-\sigma_0)\varepsilon_1^0 - \cos(\sigma-\sigma_0)\varepsilon_3^0\}\\
\frac{d\varepsilon_3^0}{d\sigma} &= +\frac{\lambda(\sigma)}{2}\{\cos(\sigma-\sigma_0)\varepsilon_2^0 - \sin(\sigma-\sigma_0)\varepsilon_4^0\}\\
\frac{d\varepsilon_4^0}{d\sigma} &= +\frac{\lambda(\sigma)}{2}\{\cos(\sigma-\sigma_0)\varepsilon_1^0 + \sin(\sigma-\sigma_0)\varepsilon_3^0\}
\end{aligned}$$

These equations should be integrated, taking into account the relations:

$$\begin{aligned}
\lambda(\sigma) &= \frac{1}{\zeta_3^4\,\hat{s}^3} f_{py}\\
\hat{s} &= 1 + \zeta_1\cos\sigma + \zeta_2\sin\sigma\\
z &= \frac{1}{r} = \zeta_3^2\,\{1 + \zeta_1\cos\sigma + \zeta_2\sin\sigma\}\\
\frac{dr}{d\tau} &= \zeta_3(\zeta_1\sin\sigma - \zeta_2\cos\sigma)\\
\chi &= \frac{\sigma-\sigma_0}{2}
\end{aligned}$$

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_3\\ \varepsilon_2\\ \varepsilon_4\end{pmatrix} = \begin{pmatrix}\cos\chi & \sin\chi & 0 & 0\\ -\sin\chi & \cos\chi & 0 & 0\\ 0 & 0 & \cos\chi & -\sin\chi\\ 0 & 0 & \sin\chi & \cos\chi\end{pmatrix}\begin{pmatrix}\varepsilon_1^0\\ \varepsilon_3^0\\ \varepsilon_2^0\\ \varepsilon_4^0\end{pmatrix}$$

Here (f_{px}, f_{py}, f_{pz}) are the non-dimensional components of the perturbing force acting upon the body. The integration must start from the appropriate initial conditions at $\sigma = \sigma_0$ ($\tau = 0$).

The initial conditions should be obtained from the initial values $(\vec{\boldsymbol{r}}_0,\,\vec{\boldsymbol{v}}_0)$ of the position and velocity of the body. In particular σ_0 is the true anomaly of the initial position at the initial osculating orbit; the initial values of $(\varepsilon_1^0, \varepsilon_3^0, \varepsilon_2^0, \varepsilon_4^0)$ are obtained directly from the orbital frame at perigee of the initial osculating orbit. The other initial values are:

$$\text{at}\quad \sigma = \sigma_0: \quad \tau = 0, \quad \zeta_1 = e_0, \quad \zeta_2 = 0, \quad \zeta_3 = \frac{\sqrt{\mu|\,\vec{\boldsymbol{r}}_0|}}{|\,\vec{\boldsymbol{r}}_0 \times \vec{\boldsymbol{v}}_0|}$$

where e_0 is the eccentricity of the initial osculating orbit and μ the gravitational constant of the attractive center (the Sun in heliocentric orbits).

The mains characteristics of DROMO are:

- Unique formulation for the three types of orbits: elliptic, parabolic and hyperbolic. So, the singularity that appears in the proximity of parabolic motion when using different formulations for elliptic and hyperbolic orbits disappears.
- It uses orbital elements as generalized coordinates (as the Lagrange's Planetary equations); as consequence, the truncation error vanishes in the unperturbed problem and is scaled by the perturbation itself in the perturbed one. The method doesn't have singularities for small inclination and/or small eccentricities, unlike the Lagrange's planetary equations. The orbital plane attitude is determined by Euler parameters which are free of singularities.
- The use of Euler parameters gives easy auto-correction as well as robustness. The error propagation shows better performances than in the cases of Cowell's or Encke's methods. Easy programming, since they use the components of perturbation forces in the orbital frame. This makes easy the use of models proper of Orbital Dynamics.
- A precise and fast simulator is obtained by using this method with variable step routines with effective step control, as Runge-Kutta-Fehlberg or Dormand-Prince types. However, routines with fixed step can be used also without reduction in performances. Multistep routines —like the classical one of Shampine & Gordon [36] (DE)— can be used also. In fact, this kind of routines show excellent characteristics because, from a practical point of view, keep the accuracy and reduce the number of function calls significantly.
- It is not necessary to solve Kepler's equation in the elliptic case, nor the equivalent for hyperbolic and parabolic cases, since time is one of the dependent variables determined by the method itself.

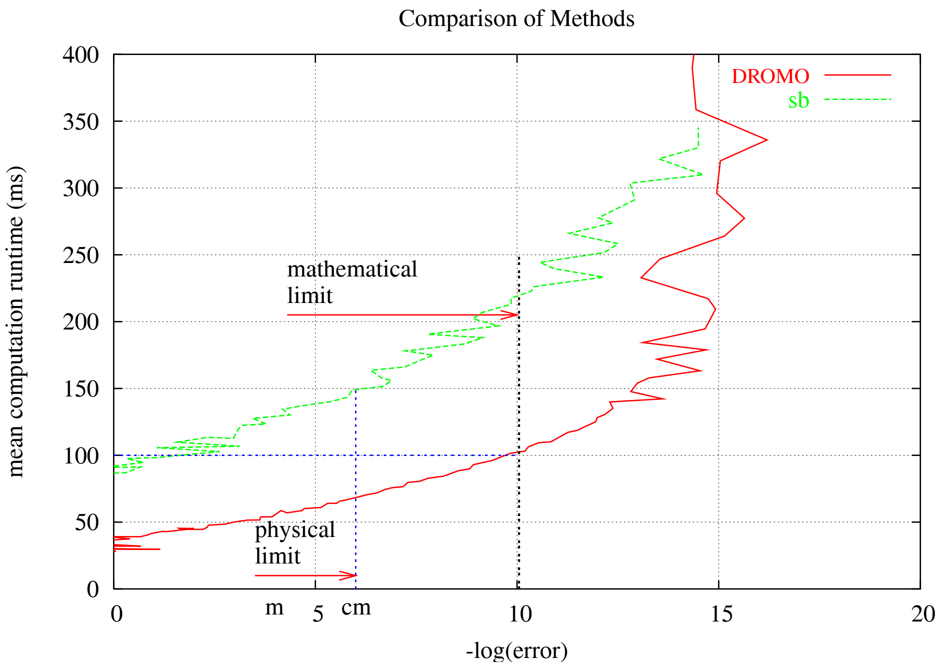

Figure 2: Comparison between DROMO and Sperling-Bürdet

In terms of *accuracy* DROMO with a classical Runge-Kutta-Fehlberg routine of order 7 —RKF7(8)— turn out to be one of the best combinations than can be used for high fidelity propagation of orbits. In [12] we shown that DROMO beats the Sperling–Bürdet's method solving the *example 2b* of the famous book of Stiefel and Scheifele: [9]; such an example has been used also in other works (for example, in the book [37]).

In that comparison the *exact solution* is not the one given in the book [9]. Instead, we recalculated the solution two times using both propagators with the maximum accuracy; we took as *exact solution* the common part obtained in both calculations.

The computations have been done: 1) in the same computer (Intel Xeon 3056 MHz microprocessor, 2 Gb RAM), 2) with the same compiler (Intel C++ 8.1.022), 3) with the same integrating algorithm (Runge-Kutta-Fehlberg (RKF) 7(8) of variable step-size better than the RKF 4(5) taken from the [38]), and 4) in the same computer conditions (processor load, etc). Moreover, to minimize the effect of uncontrolled factors on the computation time we have repeated the former task 30 times and we have obtained the mean value of runtime.

Figure 2 which has been taken from Ref. [12] shows the results of the comparison. The mean computation runtime is plotted in ordinates and the common logarithm of the norm of the error vector ($-\log(|\Delta\bar{x}|)$ in abscissas. This last quantity is a measure of the quality of the solution: it is approximately equivalent to the number of exact decimal digits of the solution plus one.

The plot shows better performances for DROMO; it seems to be quicker for the same precision, or equivalently, it seems to be more accurate for identical computational time.

These differences are mainly due to the lower order of DROMO (8 ODE's) compared with the Sperling-Bürdet's method (13 ODE's). But there are other reasons also: in the Sperling-Bürdet's method the calculation of the "second members" of equations requires to process perturbation forces through numerical treatments of some length; this also happens in similar methods based on regularization techniques as the KS's method. In our method however, forces hardly require manipulation. Note that the right hand sides of equations only include their components in the orbital frame, which are obtained by simple scalar products. Moreover, the simplicity of programming, joined to the clearness and the simplicity of equations governing the evolution of Euler parameters, strengthens our conviction in the method's advantages.

In [39] a comparison between DROMO and Cowell's method have been carried out in the field of interplanetary trajectories; the idea was to evaluate the accuracy of both methods calculating the post-trajectory of a spacecraft when a planet fly-by is included in the mission. Figure 5 shows the results of two simulations performed in that paper. In that comparison, and regarding the numerical integration methods, a variable step routine with effective step control has been programmed. In particular, the chosen scheme is an 8th order embedded Runge-Kutta method implemented by Dormand-Prince (DOPRI853). This method also allows an accurate dense output and the possibility of implementing events detection with a slight cost in additional function evaluations [40]. These brand-new capabilities, that are becoming strongly demanded, together with the reasonably good performance, numerical precision and easy implementation, make this type of single-step methods worth considering for applications in the field of orbital mechanics [41]. The propagation tool have been programmed in C although some combined C/FORTRAN90 programming has been necessary in order to use the SOFA routines for Earth Attitude, that are provided in FORTRAN90 by the International Astronomical Union.

The comparison of the results obtained by means of DROMO and Cowell's method for the integration of several direct transfers to outer planets in the Solar System exhibits the great differences in the final position that has been computed. This is particularly critical in the cases where gravity-assisted maneuvers are performed, since they are extremely sensible missions in which slight shifts in the conditions at the arrival to the sphere of influence of the target body may lead to completely different subsequent trajectories.

The comparison performed in [39] is not completely fair for the Cowell method. The reasoning is as follows: for any method, DROMO or Cowell's method, exist a numerical integrator that provides the best performances of the method by achieving the required numerical accuracy after propagation over some specified simulation time. The point is that these integrators need not be the same for the different methods considered. Thus, for each method, we should select the numerical integrator that minimizes the CPU time needed to achieve the specified error.

In general, the Cowell method reach its better performances when is used in conjunction with the Störmer-Cowell algorithms to integrate the equations. Thus, in [42] we performed a comparison between DROMO and the Störmer-Cowell method by using an analytical solution which appears in the well known *problem of Tsien*: a satellite perturbed by a constant radial thrust.

In the Tsien's problem and for a critical value of the perturbing thrust, there is an asymptotic motion from a circular orbit of radius R_0 to a final circular orbit of radius $2R_0$ (see figure 3). This last circle is an unstable limit cycle which can be calculated analytically but the *numerical* obtention of such an analytical solution is not easy.

A suitable measure to evaluate performance of both propagators is to calculate the number of orbits until the numerical solution starts to deviate from the asymptotic orbit. A deviation is considered, when the relative error of the numerically computed position is larger than a threshold. Here R is the current orbital radius which must be compared with the radius of the asymptotic orbit $2R_0$.

$$\frac{|2R_0 - R|}{2R_0} < 10^{-3}$$

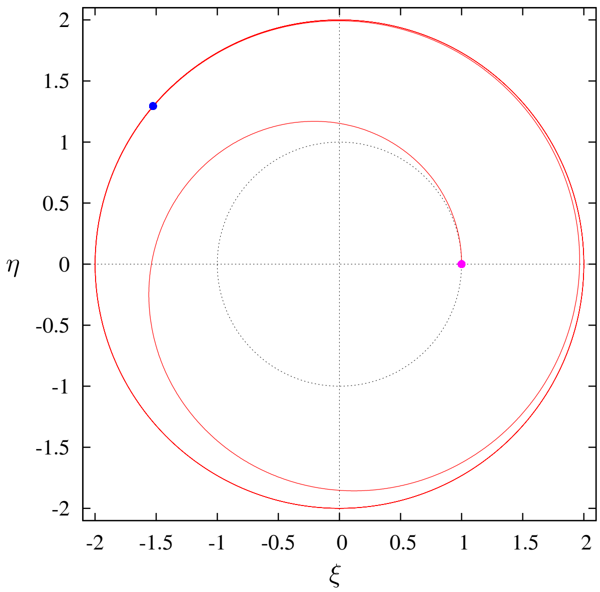

Figure 3: Satellite trajectory in the asymptotic case of the Tsien problem

To allow for a fair comparison, integrators of same order are used for the DROMO and Störmer-Cowell formulations. For DROMO, the integrators of the Runge-Kutta-Fehlberg family have proven to be very efficient and accurate. These schemes [43, 44] of order 5 to 8 are compared to Störmer-Cowell implementations [45] of equal maximum order. In addition, integrators of the multistep method of Shampine & Gordon [36] (DE 5-8) are tested and compared to Störmer-Cowell, too. The implementations for Störmer-Cowell and the DE integrators are modified to obtain a fixed order version to be compared with RKF integrators.

Figure 4 shows the number of stable orbits based on the initially given relative tolerance of the integrators. It is evident that DROMO in combination with RKF integrators has a better stability than Störmer-Cowell. However the runtime of the DROMO method is higher than of the Störmer-Cowell method of equal order. This drawback can in part be accounted for by using the DE integrator, which is faster but less accurate, for DROMO.

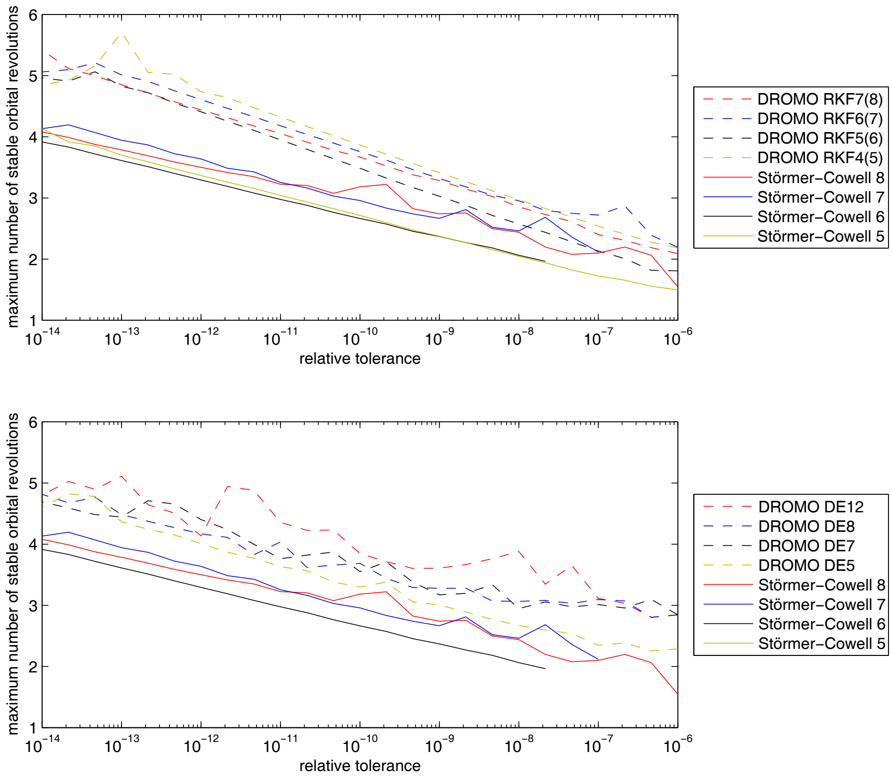

Figure 4: Comparison of method stability versus relative tolerance

In order to evaluate the computational cost of the different methods under equal conditions, the integrators have to be tuned to a similar performance. Therefore a common integration range and accuracy is chosen for them. According to figure 4, all integrators can be stable for up to 4 orbital revolutions. For fair comparison, the relative errors are chosen in such a way, that the integrators are stable only within that specified range. DROMO RKF7(8) and RKF6(7) can achieve this with $\epsilon_{rel} = 10^{-11}$ while the equal order Störmer-Cowell propagators need a tighter tolerance of $\epsilon_{rel} = 10^{-14}$. The results show comparison only of integrators of order 7 and 8 because, for both methods, they perform significantly better in terms of runtime. The evaluation is performed 100 times and table 1 shows the mean runtime, the number of steps and the function calls. It indicates similar processing time for DROMO RKF and Störmer-Cowell of the same order even though the number of function calls of DROMO is higher. This is due to the specific characteristics of the Tsien problem. The higher number of function calls in DROMO does not influence the runtime significantly, because the calculation of the perturbations is not very costly. In the Störmer-Cowell method the runtime is influenced by the fact, that coefficients have to be recalculated for each integration step. Using DROMO formulation in combination with the multistep DE integrator requires less function calls. For these integrators the runtime are not shown because they are implemented in a different programming environment.

From the analysis carried out in [42] some conclusions can be drawn.

- In terms of *accuracy* DROMO with the Runge-Kutta-Fehlberg routine RKF7(8) turn out to be the best combination since they provide a longer and more stable description of the asymptotic orbit.

Method	DROMO RKF7(8)	DROMO RKF6(7)	DROMO DE8	DROMO DE7	SC 8th order	SC 7th order
Rel tolerance	1e-11	1e-11	1e-11	1e-12	1e-14	1e-14
Runtime in s	0.21	0.47	-	-	0.24	0.24
Function calls	2379	4850	1113	1623	439	536
Number of steps	181	483	-	-	431	529

Table 1: Runtime comparison for 4 complete orbits

- In terms of *function calls* the Störmer-Cowell formulations turns out to be the best formulation since it provides the lower number of call to the derivative functions.

Due to the plus of accuracy provided by the DROMO formulation, this scheme is the most appropriated for the propagation of orbits when a high-fidelity description of the trajectory is mandatory. This plus of accuracy, however, has a cost: the higher number of function calls due to the Runge-Kutta-Fehlberg routine used to perform the integration.

However, and from a *global point of view*, the combination of DROMO with the multistep method of Shampine & Gordon [36] (DE) shows excellent characteristics because: 1) the accuracy worsens in a small amount, relative to the accuracy provided by the combination DROMO + RKF7(8), and 2) the number of function calls reduce in a significant way. Regarding this last point, it should be noticed that the Störmer-Cowell formulas requires one function call per step, and the multistep method of Shampine & Gordon [36] (DE) requires two function call per step due to the second evaluation that takes place in the *correction* part of the algorithm.

ATTITUDE PROPAGATION

In recent years a considerable amount of effort has been devoted to the development of a comprehensive theory that it provides a deeper insight into the complex dynamic behavior involved in the motion of rotating rigid bodies. Some of these efforts have been focused on alternatives ways of describing the kinematics of this motion (see for instance the complete survey [46] by M.D. Shuster); some other have been focused on the dynamics. From the kinematic point of view, one has a certain degree of freedom, since the rotation matrix which determines the relative orientation between two reference frames can be parameterized in more than one way. By having available several different approach for viewing the kinematics, more insight can be gained into a specific problem. In general, the best approach is clearly problem dependent. The most commonly used parameterizations for the attitude kinematics are the Eulerian angles, the Euler-Rodrigues parameters (quaternion formulation), the Andoyer variables, the Cayley-Klein parameters and the Cayley-Rodrigues parameters (see [46, 47, 48, 49, 50, 51]).

The dynamics of the rotational motion may be deduced from the angular momentum equation which describes the influence of the external torques on the attitude of the rigid body, through the angular velocity concept. Basically, the rotational dynamics of a rigid body is an initial value problem: for a given orientation and its change rate with respect to an inertial frame at an initial time and the force acting on it, find its attitude at any instant. From this point of view, Euler equations of motion provide a complete and well-defined framework. However, the complete analytical solution of this system of three nonlinear, coupled differential equations is still unknown in the general case. Special cases for which solutions have been found include the torque-free motion (Euler-Poinsot case) and the motion of the symmetric top forced by gravitation (Lagrange case). The existence of analytical solutions for such special cases cherished hopes about the existence of a general analytical solution which should be discovered some day. However, deeper analysis shown later that this hope it was just wishful thinking. In fact, a complete description of the rotational motion of a rigid body turned out to be a formidable task; many of the most prominent mathematicians of our time failed in their attempt to solve this problem (see [52]).

DROMO is based on a method which combines regularization, linearization and perturbations techniques and it turns out to be clearly advantageous when it is compared to other traditional methods. A question arises here in a natural way: is it possible to use a similar technique with the attitude dynamics of a celestial body?

The attitude motion of a rigid spacecraft reduces to the Euler-Poinsot case when all the perturbations torques are neglected. This torque-free motion can be considered as the unperturbed case and its analytical solution is very well known. The idea underneath the NEODROMO project tries to take advantage of the knowledge of the analytical solution of this *unperturbed problem* in order to solve with more accuracy the **perturbed problem**. Some effort have been made in our group GDT trying to obtain a formulation similar to the one carried out in [12] for the *Attitude Problem* of a rigid spacecraft. The goal was to obtain a global propagation tool with the ability to include, when necessary, both aspects of the Space Dynamics: the orbital motion of the center of mass and the attitude dynamics. Simultaneously, we opted for the quaternion formulation taking as generalized coordinates the Euler-Rodrigues parameters trying to follow, if possible, the trail of papers [53, 54]. **This theory has been developed in [55] for the case of an axisymmetric satellite**. Similar approaches have been considered in references [56, 57, 58] for the triaxial case. Also in [59] can be found a similar formulation; there is an important difference, however, since in [55] a perturbation technique is used that is absent in [59].

From a logical point of view, the procedure usually followed to solve the *unperturbed problem* —torque-free motion— requires the following two steps:

1. in a first step the time evolution of the coordinates (p, q, r) —in the body frame— of the angular velocity $\vec{\omega}$ of the celestial body is obtained. The triaxial case involves a greater complexity due to the nature of the solution of the unperturbed problem. The time evolution of (p, q, r) is given by the Jacobi elliptical functions. However, in the axisymmetric case the time evolution of (p, q, r) is given by simple harmonic functions.

2. in a second stage the coordinates fixing the attitude of the celestial body are obtained from the known expressions of (p, q, r). There are several options since the rotation matrix which determines the relative orientation between the body frame and an inertial frame can be parameterized by using: 1) Euler angles, 2) Andoyer variables or 3) Euler parameters.

In our opinion the Euler parameters is the best option, from a numerical point of view, due to the lack of singularities in the parametrization of the configuration space. However, when using **Euler parameters**, and from a practical point of view, the triaxial case cannot be handled since the time evolution of the Euler parameters is given by unmanageable expressions. On the contrary, the axisymmetric case can be solved easily and the time evolution of the Euler parameters is given by simple and elegant expressions which permit to develop a perturbation theory as it is shown in [55].

The key point in the NEODROMO formulation is the following: the triaxial case can be considered as a perturbed problem of an equivalent axisymmetric case. This is the main difference with the concepts developed in references [56, 57, 58] where the triaxial case is tackle directly.

In effect, let (I_1, I_2, I_3) be the principal central moments of inertia of the triaxial celestial body. The Euler equations governing the attitude dynamics of the body take the form:

$$\dot{p} + \frac{I_3 - I_2}{I_1}\, q\, r = \frac{L}{I_1} \tag{1}$$

$$\dot{q} + \frac{I_1 - I_3}{I_2}\, p\, r = \frac{M}{I_2} \tag{2}$$

$$\dot{r} + \frac{I_2 - I_1}{I_3}\, p\, q = \frac{N}{I_3} \tag{3}$$

where (L, M, N) are the coordinates, in the body frame, of the perturbation torque which is acting on the center of mass of the body. Let us assume that $|I_1 - I_2| < |I_1 - I_3|$ and $|I_1 - I_2| < |I_2 - I_3|$. By introducing the value

$$I_0 = \frac{1}{2}(I_1 + I_2) \tag{4}$$

the governing equations take the following form

$$\dot{p} + \frac{I_3 - I_0}{I_0}\, q\, r = \frac{1}{I_1}\{L + \tilde{L}\, q\, r\} \tag{5}$$

$$\dot{q} + \frac{I_0 - I_3}{I_0}\, p\, r = \frac{1}{I_2}\{M + \tilde{M}\, p\, r\} \tag{6}$$

$$\dot{r} = \frac{1}{I_3}\{N + \tilde{N}\, p\, q\} \tag{7}$$

where the values $(\tilde{L}, \tilde{M}, \tilde{N})$ are given by:

$$\tilde{L} = \tilde{M} = \frac{(I_2 - I_1)(I_1 + I_2 - I_3)}{I_1 + I_2}, \quad \tilde{N} = I_1 - I_2 \tag{8}$$

Thus, the governing equations (4-8) turn out to be a **perturbed problem** of an *equivalent axisymmetric body* which includes an additional perturbation torque, due to the triaxiality of the body, which has the following coordinates

$$(\tilde{L}\, q\, r, \tilde{M}\, p\, r, \tilde{N}\, p\, q)$$

in the body frame. Obviously, when $I_1 = I_2$ this perturbing torque vanishes, since $\tilde{L} = \tilde{M} = \tilde{N} = 0$ in such a case.

From a numerical point of view, this approach is quite convenient for the propagation of the attitude motion of NEO's since in many cases two principal inertia moments take similar values and the perturbation torque due to the triaxiality of the body is small. But this numerical scheme, can be used also with bodies strongly triaxials, with a slight deterioration in performances.

ORBIT DETERMINATION

One of the main goals involved in the accurate determination of NEO's orbits is the reliable assessment of the real risk of a collision between a potentially hazardous asteroid and the Earth. Orbit determination and long time span forward propagation of NEOs is particularly difficult, mainly due to the large amount of uncertainties involved when compared to artificial satellites. These difficulties advocate for a 100-year time horizon for routine impact monitoring in space surveillance issues, since the analysis of impact possibilities further in the future is strongly dependent on the action of the Yarkovsky effect, which raises new challenges in the careful assessment of longer term impact hazards [60]. We must notice that the Yarkovsky effect is the largest single source of uncertainty in trajectory predictions of < 2 km diameter asteroids [61].

Additionally, even for NEOs with very accurately determined orbits, a future close approach to another body would scatter the possible trajectories to the extent that the problem becomes like that of a newly discovered asteroid with a weakly determined orbit. Actually, both orbital integrations and direct measurements predict Yarkovsky-induced position offsets of millions of kilometers on timescales of decades to centuries when coupled with such close planetary encounters [61].

According to [60], *if the scattering takes place late enough so that the target plane uncertainty prevails over Yarkovsky accelerations, then the asteroid's thermal properties, typically unknown, play a major role in the impact assessment. In contrast, if the strong planetary interaction takes place sooner, while the Yarkovsky dispersion is still relatively small, then precise modeling of the non-gravitational acceleration may be unnecessary*.

This sensitivity to close encounters suggests the use of regularized formulations —such as DROMO or similar schemes as the reviewed in Ref. [10]— in order to stretch the uncertainties related to orbit propagation from the point of view of numerical error accumulation. The importance of the close encounters in the real NEO's orbits is crucial. Thus, in some comparisons performed between different propagation schemes (see [10]), the number of encounters involved in the NEO's trajectory to be propagated plays a significant role.

We should underline one of the strong points of DROMO: its ability to manage close encounters in a better way than other perturbation methods. Figure 5 —which has been taken from Ref. [39]— shows the heliocentric post-encounter trajectories of a spacecraft, in Uranus and Neptune respectively, calculated with the Cowell

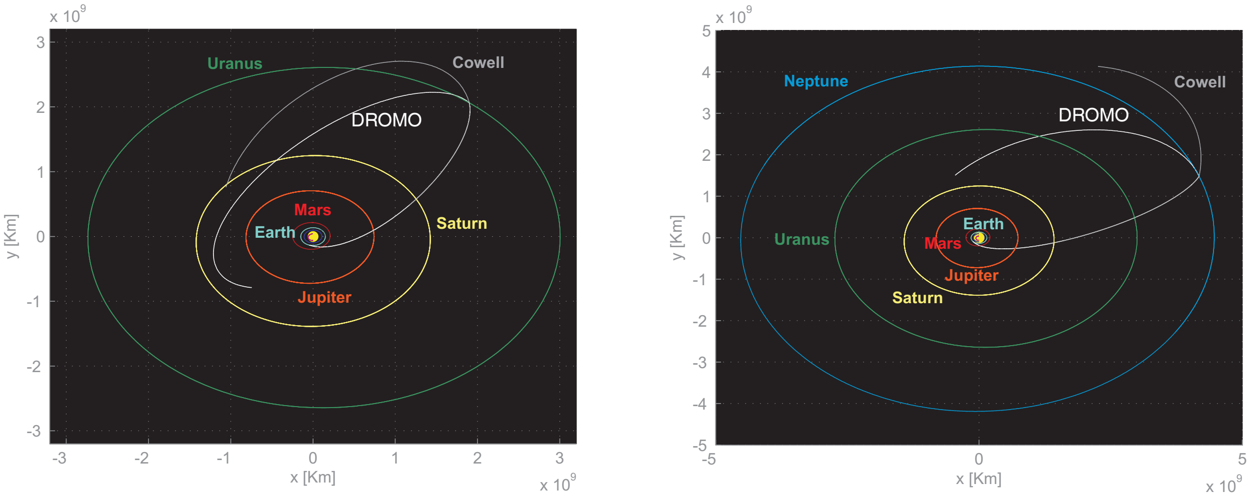

Figure 5: Close encounters with Uranus and Neptune

method and DROMO; the same numerical algorithm —the Dormand-Prince routine DOPRI853 with dense output— have been used to propagate the non-Keplerian orbits in both cases. The post-encounter trajectories are quite different due basically to the error accumulated in the Cowell propagation. Due to this differences, the entry point to the planet's sphere of influence and the entry spacecraft velocities are different in both propagations. These small differences are strongly amplified in the later encounter and give place to quite different post-encounter trajectories.

According with [62]: *the main limit to the time span of a numerical integration of the planetary orbits is no longer set by the availability of computer resources, but rather by the accumulation of the integration error. By the latter we mean the difference between the computed orbit and the dynamical behaviour of the real physical system, whatever the causes.* It is a well known fact that propagation error is exponential in the Cowell method (see [37]); this exponential growth of the errors, should be reduced as much as possible in order to increase the accuracy of the long term propagation of NEO's. For reasons that are not clear for us, the error propagation of DROMO is lower than in other formulations.

In order to put some kind of approximate bound to these behaviours, we could consider that Yarkovsky semimajor-axis drifts of the order $\sim$ 10 km per 10 years becomes crucial for an accurate orbit determination and even for estimates of an impact hazard [63]. Especially, when the calculation of an impact probability depends on the fact, if the asteroid misses of hits a phase-space "keyhole", which is much smaller than the diameter of the Earth.

To assess the possibility of impacts taking into account the Yarkovsky effect with its full uncertainty, Monte Carlo methods are usually employed. These methods are based on construction of a set of possible orbits (by applying small variations to the initial conditions) which represent the region of possible motions [64].

The accuracy of a NEO trajectory prediction depends on the fraction of the orbit sampled by astrometry, the accuracy and precision of those measurements, the interval between the time of measurement and time of prediction, and the dynamics of the model used to propagate the non-linear equations of motion [34, 64]. The Standard Dynamical Model, used for routine asteroid solutions and propagations, includes n-body relativistic gravitational forces caused by the Sun, planets, Moon, Ceres, Pallas, and Vesta, but no single non-gravitational perturbation is considered, a matter that has recently brought the debate to use an Extended Dynamical Model instead.

A meaningful example of the convenience of considering the Yarkovsky effect is that of asteroid Apophis, whose uncertainty in accelerations related to solar radiation can cause, according to [34], between 82 and 4720 Earth-radii of trajectory change relative to the Standard Dynamical Model by its 2036 Earth encounter.

In the NEODROMO project the propagation of the attitude of the NEO's is performed by the numerical scheme described in page 10. This scheme is based in a perturbation method for which the unperturbed motion corresponds to an equivalent axisymmetric body. For a real axisymmetric body the state variables are "constants" of the unperturbed solution. For a slightly triaxial body the state variables are '"quasi-constants" of the unperturbed motion.

The NEODROMO project will be able, as a consequence, to simulate the *full-two-body problem* under the action of different perturbations. In general, the numerical simulation of the *full-two-body problem* introduces stiffness in the integration, due to the differences between the characteristic times associated with the attitude dynamics and the orbital dynamics. However, due to the perturbation scheme used in the propagation of the attitude, in the case of NEO's affected by the YORP effect, the stiffness is practically removed for axisymmetric bodies and it is substantially decreased for quasi-axisymmetric asteroids. For the triaxial case, the stiffness of the problem should be taken into account by selecting an appropriated integration algorithm.

Finally we should underline that the numerical simulation of the *full-two-body problem* allows to introduce a plus of accuracy in the basic problem of determine the orbit of a NEO. Thus, some parameters used to model the dynamics of the body —for example, direction of the spin axis in the body frame— could be determined inside the level of accuracy permitted by the observations.

CONCLUSIONS

In this paper we reviewed the different elements involved in the NEODROMO project:

- a special perturbation method capable to propagate with accuracy the complex orbits of the center of mass of any celestial body. It opens a new approach in the field of NEO's dynamics due to its many good qualities and, in particular, its robustness.
- an additional propagator of the attitude of any celestial body. It is based on a perturbation scheme in which the unperturbed dynamics is selected as the motion of an equivalent axisymmetric body. The perturbation scheme is described, from a kinematic point of view, in terms of Euler parameters (a quaternion), and turns out to be more accurate than other classical and sophisticated schemes based on Hamiltonian dynamics.
- both propagators can be used together in order to simulate the *full-two-body problem* from a numerical point of view. The full simulation of a triaxial body introduces stiffness in the integration. Due to the perturbation scheme used in the propagation of the attitude, in the case of NEO's affected by the YORP effect, the stiffness is practically removed for axisymmetric bodies and it is substantially decreased for quasi-axisymmetric asteroids.
- in future editions of this meeting we will present some interesting simulations performed inside the NEODROMO project.

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